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TOPICS IN THE STUDY OF INTERCONNECTED SYSTEMS

By

E.F. Mageirou

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This report deals with the stability, stabilization and optimization of systems which can be meaningfully viewed as an interconnection of smaller subsystems. We consider systems whose properties can be determined by examining the corresponding properties of the subsystems and their interconnections rather than the system as a whole. The relevance of such systems to any theory of large scale systems is evident. NEXT PG.

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Heuristically speaking, a weak interconnection of dynamical systems should be analyzable in such a segmented fashion. We present some formulations of the weak interconnection concept in Section ~~II~~ 2 in the context of stability studies of economic models. It is shown how the modelling of an economy requires the consideration of systems formed by suitably bounded interconnections of stable subsystems. Lyapunov's second method is employed to unify numerous results as well as to provide interesting extensions.

In contrast to Section II where weakly interconnected stable systems follow from modelling requirements, Section IV considers the interconnection of stable dynamic systems through additive terms and investigates under what conditions on these additive terms stability will be preserved. The methods of dissipative system theory are used to derive such conditions, the attractive feature of which is that they can be used to design decentralized feedback stabilizing controllers.

The problem of decentralized stabilization is reduced to an infinite duration linear quadratic game, a complete solution of which is given in Section ~~V~~ 5. As expected, Riccati-type equations play an important role in such problems. This role is clarified in Section ~~V~~ 5.

Section ~~III~~ 3 deals with the comparison of the solutions of optimal control problems for interconnected systems. The theory of dynamic programming and the Hamilton Jacobi Bellman equation is the main tool used. In addition, the same theory is applied to an extensive study of optimization algorithms, the main result being the extension of the applicability of the well-known Kleinman algorithm for the Riccati equation to cover the ones encountered in Sections ~~IV~~ 4 and ~~V~~ 5. This result provides an important ingredient for any practical application of the decentralized stabilization procedures proposed in these Sections.

END OF ABSTRACT

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In contrast to Section II where weakly interconnected stable systems follow from modelling requirements, Section IV considers the interconnection of stable dynamic systems through additive terms and investigates under what conditions on these additive terms stability will be preserved. The methods of dissipative system theory are used to derive such conditions, the attractive feature of which is that they can be used to design decentralized feedback stabilizing controllers.

The problem of decentralized stabilization is reduced to an infinite duration linear quadratic game, a complete solution of which is given in Section V. As expected, Riccati-type equations play an important role in

such problems. This role is clarified in Section V.

Section III deals with the comparison of the solutions of optimal control problems for interconnected systems. The theory of dynamic programming and the Hamilton Jacobi Bellman equation is the main tool used. In addition, the same theory is applied to an extensive study of optimization algorithms, the main result being the extension of the applicability of the well-known Kleinman algorithm for the Riccati equation to cover the ones encountered in Sections IV and V. This result provides an important ingredient for any practical application of the decentralized stabilization procedures proposed in these Sections.

TABLE OF CONTENTS

Table of Contents continued

I. SCOPE OF THE PAPER

This paper examines some problems arising in the analysis of dynamical systems whose large dimensionality makes the application of standard analysis techniques infeasible due to excessive computational requirements. Such a system will be referred to as a large scale system. A theory of large scale systems (l. s. s. t) tries to identify classes of large systems whose properties can be efficiently determined. Its scope is thus twofold, first to isolate classes of large systems which are useful for applications and second to propose efficient algorithms for their analysis.

A large system structure often encountered in practice is that of an interconnected system, where a number of dynamical subsystems interact to form a large system. An electric power system is an example where the subsystems and the interconnections can be physically identified. It would be desirable to develop methods for determining the properties of the interconnected system by somehow aggregating the corresponding properties of the subsystems. Such a task perhaps too ambitious as the effect of the interconnections can be very hard to assess. It might be claimed, from a heuristic point of view, that of the subsystems are interconnected weakly it would be possible to study the whole system by aggregating the properties of the subsystems.

It is not obvious how one should define the term "weak interconnection." To define a system as weakly interconnected when its analysis can be done by aggregating its subsystem properties would be circular. In Sections II and IV we study some non-circular formulations

of this concept as applied to the stability and stabilization of interconnected systems. An examination of the meaning of economic stability carried out in Section II results in the concept of Metzler stability or stability independent of adjustment rates: Economic processes cannot be adequately modelled by a single differential equation for the simple reason that the rates at which prices adjust to a disequilibrium are varying in an unpredictable way. It is meaningful to refer to an economic model as stable if this property holds independent of adjustment rates. The implications of Metzler stability are explored in Section II by Lyapunov methods which provide a unified setting for known results on linear systems as well as their generalization to nonlinear ones. Most of the conditions thus derived can be interpreted as requiring subsystem stability plus conditions on the interconnections, usually stipulating a small "intensity" of interconnection. We thus obtain a class of interesting formulations of the weak interconnection concept.

The concept of a stable system formed by a weak interconnection of stable subsystems is shown in Section II to be a consequence of the economic modelling requirements. But such a system can arise from different considerations. It is often the case, as in a large power-system, that the subsystem interconnections might fail thus isolating particular subsystems. It is thus desirable that each subsystem have the capacity to operate successfully in isolation which means among other things, that each subsystem must be stable.

In Section IV we consider an explicit model for such systems, namely differential systems of the form

$$\dot{x}_i = f_i(x_i, t) + h_i(x_1, \ldots, x_N; t) \qquad i = 1, 2, \ldots, N .$$

The isolated systems

$$\dot{x}_i = f_i(x_i, t) \qquad i = 1, 2, \ldots, N$$

interact through the additive interconnections h_i. Starting with the assumption of stability of the isolated systems, we consider the stabilizing or destabilizing effects of the h_i's. The concept of dissipative systems of J. C. Willems plays an important role in Section IV as it allows under "weak interconnection" conditions on the h_i the construction of a system Lyapunov function from subsystem ones. Hence the stability or instability of a system can be obtained from subsystem considerations.

The design of stabilizing controllers for interconnected systems can be based on the above theory and is developed in the latter part of Section IV. In designing a control scheme for a system above elements are in distinct locations it cannot be assumed that system information (measurements) is transmitted to a central controller without a cost. Under the plausible assumption that a centralized control scheme is more effective than a decentralized one, it becomes important to balance information transmission cost and control benefit. Such a problem being hard even to express analytically, it is customary to consider the extreme cases of a centralized versus a decentralized scheme. In particular Section IV considers the design of a decentralized linear feedback control scheme where one controller is associated with each subsystem $\dot{x}_i = f_i(x_i, t; u_i)$. The control u_i can be a linear function of the i-th subsystem measurements only, that being either the state x_i or an output $y_i = C_i x_i$.

The stabilization policy proposed is to render each subsystem dissipative enough for the interconnection terms to be insufficient to alter the stability of the system. The important result of Section IV is the development of algorithmic procedures for determining both whether stabilization is possible and, if it is, to generate the appropriate linear subsystem feedback. This is done by showing that transforming a subsystem into a dissipative one through linear feedback is possible if and only if a certain differential game has a finite value over closed loop strategies. The differential game considered has linear dynamics, a quadratic value functional and is of infinite duration. An analysis of such games, which are interesting in their own right, is presented in Section V. The interesting result is that the existence of a finite value is equivalent to a certain algebraic equation (of the Riccati type) having a positive definite solution. The final outcome is the reduction of the stabilization problem to solving an algebraic equation for each subsystem, very much in the spirit of the goals of l. s. s. t. set in the beginning of this section.

To complete the stabilization scheme of Section IV, some computational matters are considered in Section III. The solution of a Riccati equation

$$A'K + KA + Q = KBB'K \qquad Q \geq 0$$

can be carried out by an iterative procedure known as Kleinman's algorithm (though being a special case of an approximation in policy-space procedure due to Bellman). The stabilization problem of Section IV can be reduced to a similar equation with the matrix Q being indefinite. The

main result of the latter part of Section III is to show that Kleinman's algorithm converges for an arbitrary Q, provided of course that a solution exists. The proof of the result relies on a dynamic programming approach based on inequalities of the Hamilton-Jacobi-Bellman type which are treated in Section III. Some relevant results on Riccati equations have been collected in Appendix 1 and some examples of an implementation of the algorithm on a digital computer are presented in Appendix 2.

Finally, the dynamic programming approach of the HJB inequalities is applied to the comparison of optimal systems. Section III contains results comparing the optimal payoffs of two systems which are particularly useful when one system is decoupled (and hence easy to optimize), the other system resulting from the first by introducing additive coupling terms. Some estimates of the performance of the optimal decoupled system policy when applied to the coupled one are also provided.

II. METZLER STABILITY

1. Introduction

In this section we intend to survey and unify the work of several authors on stability-type questions pertaining to the modelling of societal, mainly economic systems. Until relatively recently, economic theory was interested in comparative statics, that is the comparison of possible equilibria, but not in their stability. In fact, in many economic writings there was talk of equilibria without mention of any specific dynamical system. It was understood that the relevant dynamical systems was the "market" and its law was that of supply and demand. This system was supposed to have a unique, stable equilibrium point and if the equilibrium were to change because of exogeneous reasons, the system would swiftly adjust to its new equilibrium.

Economists did not pursue a rigorous study of the above assumptions despite the fact that the actual economy seldom appeared to reach its equilibrium. It was Samuelson [10] who first pointed out the need of studying the economy as a dynamical system. This was done in the context of showing that a set of conditions derived on the basis of plausibility arguments by the well known economist Hicks were neither necessary nor sufficient for the stability of dynamical systems.

The reluctance of economic theory to postulate dynamic models is quite justifiable. The adjustment processes of economic activity cannot be observed accurately enough in order to form the basis of a dynamical theory. Furthermore, economic theory can't be reasonably

expected to predict rates of adjustment but rather the direction of adjustment: It can be predicted that prices will rise in response to excess demand but the rate of price increase might very well depend on non-measurable factors and lies outside the domain of the theory. This inability to specify adjustment rates is an inherent feature of economic dynamic models. The important consequence is that a model can be said to imply a certain property only if that property holds independent of the adjustment rates. A property must hold in every model in a rather large class of models before it can be claimed to be a property of the economic system being modelled.

The goal of this section is to study stability theory in the above context. We will examine dynamical systems of the form

$$\frac{d}{dt} x = K(t)\, f(x, t) \; ; \qquad K(t) \in \mathscr{K} \tag{2.1}$$

where x is a n-dimensional state vector, $f : R^n \times [t_0, \infty) \to R^n$ a given function, $K(t)$ a positive definite diagonal matrix. The function f incorporates the theoretical specification of the situation. On the other hand the matrix functions $K(t)$ belong to a class of functions $\mathscr{K}$ and model the theoretically unspecifiable adjustment rates. Of course certain restrictions must hold on f and $\mathscr{K}$ so that solutions to (2.1) exist and are unique. We assume throughout that such conditions have been imposed. A collection of systems (2.1) is said to possess a stable equilibrium point x^* if x^* is a globally asymptotically stable equilibrium point for every $K(t) \in \mathscr{K}$. We shall refer to such an equilibrium point as a Metzler-stable one (with respect to $\mathscr{K}$). Metzler stability will be studied for various classes of functions $f(x, t)$ and

adjustment rates $\mathcal{K}$. Finally, the relation of the Metzler stability condition to recent work on the stability of "interconnected systems" as well as to the "aggregation" of large scale systems will be examined.

2. The Hicks Conditions in Linear System Stability

2.1. The Hicks Conditions

The process of exchange of n commodities in a market system is accomplished through the price mechanism: At prices $p_1, p_2, \ldots, p_n$ a quantity $D_i(p_1, \ldots, p_n)$ will be demanded of the i-th commodity and a quantity $S_i(p_1, \ldots, p_n)$ will be supplied. The prices $p_1^*, \ldots, p_n^*$ such that $D_i(p_1^*, \ldots, p_n^*) = S_i(p_1^*, \ldots, p_n^*)$ are called equilibrium prices because as long as they prevail the exchange process satisfies the economic needs of all its participants. It is of fundamental importance to economic theory to show that the equilibrium price is stable, i.e. that there are economic forces that guide the prices towards the equilibrium price.

A special case of the above problem was dealt with in the works of the early economists Walras, Marshall et al. They considered the case where all the prices except the i-th one were kept constant by fiat; only the dynamics of the i-th price p_i were considered. It is reasonable to assume that a price $\overline{p}_i$ exists for which demand equals supply for the i-th commodity, the remaining prices being kept at their preassigned values. Economic reasoning stipulates that for $p_i < \overline{p}_i$ there will be an excess demand for the i-th commodity which will lead the economic

agents controlling p_i to increase it. The opposite happens of $p_i > \bar{p}_i$. Therefore, if all prices except one are kept fixed, the market will lead the remaining price to its equilibrium point.

It should be noted that the equilibrium price $\bar{p}_i$ depends on the prices p_j, $j \neq i$, which have been kept constant during the adjustment of the i-th market. After the adjustment is over, there is no guarantee that the remaining markets will be in equilibrium, i.e. there will be no excess demand or supply in the market for the j-th commodity $(j \neq i)$. Further economic adjustments will be needed and the Walras-Marshal theory can make no statements about their convergence or not to a price equilibrium.

In retrospect it seems obvious that a model which allows one price to adjust while the others are kept fixed cannot be very realistic. A model for price adjustment must consider a process in which all prices adjust simultaneously. This is consistent with our perception of the operation of the economy. However, models for such processes and analyses of their stability did not appear until the British economist J. Hicks in his book Value and Capital attacked this problem as follows: The Walras-Marshal formulation considers a change in the i-th price, the other prices being kept constant, and postulates that a price p_i above the equilibrium results in the supply exceeding demand for the i-th commodity and vice versa. Hicks' formulation is more complicated. He assumes that the equilibrium $p_1^*, \ldots, p_n^*$ has been established and considers the effect of a change in the i-th price from its equilibrium position p_i^*. However, the remaining prices are not kept fixed but are assumed to adjust so as to restore equilibrium in

their respective markets. Specifically, he defines a market for n-commodities to be perfectly stable if the following conditions hold [3, p. 248]: "A rise in price of any commodity must make the supply of that commodity exceed the demand (under any of the following circumstances): (a) if all other prices are given, (b) if some other prices are adjusted so as to preserve equality between demand and supply (of the corresponding commodities), (c) if all other prices are so adjusted." In a perfectly stable market an increase of the i-th price over the equilibrium price p_i^* will cause excess supply of the i-th commodity even after the other prices have adjusted. Economic forces will therefore move the i-th price towards its equilibrium.

Although the requirements for perfect stability might be natural for an economist, it is not clear what their implications are for a dynamic adjustment process. It is worth noting that Hicks did not postulate his conditions with reference to any specific dynamic adjustment process. Samuelson [11] studied perfectly stable markets and dynamic adjustment processes and showed that there need not be a relation between perfect stability and the convergence of the adjustment processes to an equilibrium. To see this, write the excess demand for the i-th commodity as

$$E_i(p_1, \ldots, p_n) = D_i(p_1, \ldots, p_n) - S_i(p_1, \ldots, p_n) \tag{2.2}$$

From the definition of p_i^*, $E_i(p_1^*, \ldots, p_n^*) = 0$, $i = 1, \ldots, n$. Consider now the Jacobian of $E(\underline{p})$ at p^*

$$\frac{\partial E}{\partial \underline{p}} = \{a_{ij}\} = \frac{\partial E_i}{\partial p_j}\Big|_{p_1^*, \ldots, p_n^*}$$

It can be shown by looking at small Δp_i that the perfect stability condition imply [3, p. 315] that the principal minors of $\{a_{ij}\}$ must alternate in sign, i.e.,

$$a_{ii} < 0, \quad \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} > 0 , \quad \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix} < 0 \quad \text{etc.} \tag{2.3}$$

Now, a reasonable model of an adjustment process is

$$\frac{dp_i}{dt} = k_i\, E_i(p_1, \ldots, p_n) \qquad i = 1, \ldots, n \tag{2.4}$$

and in the case where E_i is a linear function (and $k_i = 1$)

$$E(\underline{p}) = A(\underline{p} - \underline{p}^*) \tag{2.5}$$

the system (2.4) becomes

$$\frac{d\underline{p}}{dt} = A(\underline{p} - \underline{p}^*) \tag{2.6}$$

For a linear excess demand function as in (2.5) it can be shown [3, p. 315] that the perfect stability condition is equivalent to (2.3) and hence if perfect stability implied convergence of the adjustment processes, we would have that (2.3) is a sufficient condition for A to be a stable matrix. This can be shown to be false through simple counterexamples and it is therefore possible to exhibit perfectly stable markets which are not stable in a dynamic sense.

The role of the Hicks conditions to stability was eventually clarified by Metzler [6]. He first pointed out that the stability of adjustment processes of the kind of (2.4) would depend strongly on the relative speeds of adjustment $k_1, \ldots, k_n$. Second he noted that the verbal statement of perfect stability by Hicks allows some markets to adjust instantaneously while others do not adjust at all; the role of the markets could then be reversed. Such a stability definition can only make sense if stability is independent of the speed of adjustment. In this way Metzler was led to formulate a version of what we call here a Metzler stability problem. He considers a linearized version of the adjustment processes around the equilibrium price p^*, and allows time invariant but arbitrary positive speeds of adjustment. In terms of (2.1) he considers the stability problem

$$\dot{p} = KA(p - p^*) \qquad K \in \mathscr{K}_0 = \left\{ \begin{matrix} \text{Positive definite} \\ \text{diagonal matrices} \end{matrix} \right\} \tag{2.7}$$

The Hicks conditions are related to (2.7) as follows

LEMMA (2.8) (Metzler) The Hicks conditions are necessary for the stability of p^* for all systems in (2.7).

Proof. The characteristic equation of (2.7) is

$$c(\lambda) = \lambda^n - c_1 \lambda^{n-1} + c_2 \lambda^{n-2} - \ldots + (-1)^n c_n = 0$$

where c_r is the sum of all principal minors of order r of KA, $1 \leq r \leq n$. If now $k_i = 0$ for some $i \in \{1, 2, \ldots, n\}$ the characteristic equation will be

$$c(\lambda) = \lambda\, c_1(\lambda) = 0$$

where $c_1(\lambda)$ is the characteristic polynomial corresponding to the matrix $(KA)^{ii}$ i.e., the $(n-1)\times(n-1)$ matrix which results from removing the i-th column and row from KA.

It is necessary for the stability of (2.7) for all K that $c_1(\lambda)$ has roots in the L.H.P. Otherwise we could choose a positive k_i small enough so that some root of the characteristic polynomial is in the RHP. By repeating the above argument we conclude that every principal submatrix of A must be a stable one. Now a necessary condition for a $k \times k$ matrix B to be stable is that

$$(-1)^k \det B > 0 \quad .$$

To see this let $d(\lambda)$ be the characteristic polynomial of B. Clearly, $d(\lambda) > 0$ for some real positive λ. Furthermore $d(0) = (-1)^k \det B$. If $d(0) < 0$, we could conclude the existence of a real positive root of $d(\lambda)$. Hence we must have $d(0) = (-1)^k \det B \geq 0$ if B is to be stable. The statement of the lemma follows immediately. Q.E.D.

We have actually proved a stronger condition for stability, namely

LEMMA (2.9) A necessary condition for the stability of p^* for all systems in (2.7) is that all principal submatrices of A, i.e.,

$$a_{ii} \;,\quad \begin{pmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{pmatrix} \;,\quad \begin{pmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{pmatrix} \qquad \text{etc.}$$

are stable matrices.

It can be shown through counterexamples that the Hicks conditions are not sufficient for stability of the systems in (2.5) or (2.7). In an important case though they are sufficient, the case where the off-diagonal elements of the matrix A in (2.5) are nonnegative. A matrix with this property is usually called an M-matrix. The economic interpretation of A being an M-matrix is that (2.5) refers to the market of substitute commodities, where an increase in the price of a certain commodity leads to an increased demand for all other commodities. For substitute commodities and a linear adjustment process, the Hicks conditions on A are sufficient for the stability of all systems in (2.5).

PROPOSITION (2.10) (Metzler) The Hicks conditions are necessary and sufficient for the stability of all systems in (2.5) provided A is an M-matrix.

Proof. Sufficiency: The Perron-Frobenius theory of positive matrices can be easily applied to M-matrices to show that the eigenvalue of an M-matrix with largest real part is real [4, p. 255]. Consider now the characteristic polynomial of KA

$$c(\lambda) = \lambda^n - c_1 \lambda^{n-1} + c_2 \lambda^{n-2} - \dots + (-1)^n c_n = 0 \qquad (2.11)$$

where c_r is the sum of all principal minors or order r. If the Hicks conditions hold, $c_r \cdot (-1)^r > 0$ and thus all coefficients of $c(\lambda)$ are positive. This shows that (2.11) has no roots on the positive real axis and, by the result stated in the beginning of the proof, it can have no roots with nonnegative real parts. The necessity part is Lemma (2.8).

Q. E. D.

The obvious question is to establish necessary and sufficient conditions for the stability of all systems in (2.4). The necessary condition in Lemma (2.9) is much stronger than the Hicks conditions, however it is not clear whether it is also sufficient.

2.2 Generalizations of the Metzler Stability Problem

We now want to extend the problem posed by Metzler by allowing time varying speeds of adjustment. Namely, we consider the systems

$$\left.\begin{aligned} &\frac{d}{dt}\, p = K(t)\, A(p-p^*) \\ &K(t) \in \mathscr{K}_1 = \{K(t)\text{: Diagonal matrices, continuous} \\ &\qquad\qquad \text{in } t \text{ and } 0 < k \le k_i(t)\} \end{aligned}\right\} \tag{2.12}$$

and seek conditions on A for p^* to be a stable equilibrium point for every system in (2.12). We provide next a sufficient condition for this to happen

PROPOSITION (2.13) Define $\widetilde{A}$ by

$$\widetilde{a}_{ij} = \begin{cases} a_{ij} & \text{for} \quad i = j \\ |a_{ij}| & \text{for} \quad i \neq j \end{cases}$$

If $\widetilde{A}$ is a stable matrix, all systems in (2.12) are stable.

<u>Proof</u>. By a theorem of Ostrowski [8] there exists a positive diagonal matrix D such that $D\widetilde{A}D^{-1}$ is row diagonally dominant, i.e.,

$$a_{ii} + \sum_{j \neq i} \frac{|a_{ij}|}{d_j}\, d_i \le \pi < 0 \tag{2.14}$$

for some π and all $i = 1, 2, \ldots, n$. Consider now the system $\dot{x} = K(t)Ax$. The stability of the origin for this system implies the stability of p^* in (2.12). We will proceed to show that $\dot{x} = K(t)Ax$ is stable. Let $y_i(t) = d_i x_i(t)$ and

$$y(t) = \max(|y_1(t)|, |y_2(t)|, \ldots, |y_n(t)|)$$

Furthermore define the index set $J(t)$ as

$$J(t) = \{j \quad |y_j(t)| = y(t)\} \quad .$$

The function $y(t)$ is continuous but not necessarily differentiable for every t. We will show that the right-hand derivative of $y(t)$ exists and satisfies a differential inequality from which we will derive the required stability of $\dot{x} = K(t)Ax$.

From the definition of $y(t)$ it follows that $y(t) = 0$ is equivalent to $x_i(t) = 0$. We will consider therefore the right-hand derivative of $y(t)$ whenever $y(t) \neq 0$. Consider the expression

$$D(\Delta t) = \frac{y(t+\Delta t) - y(t)}{\Delta t} \quad ; \qquad \Delta t > 0 \quad , \qquad y(t) \neq 0$$

Since $y(t) \neq 0$ it follows that the $|y_j(t)|$ $j \in J(t)$ are differentiable at t. Furthermore if $d/dt\, |y_k(t)| > d/dt\, |y_\ell(t)|$ for $k, \ell \in J(t)$, $|y_k(\tau)| \geq |y_\ell(\tau)|$ for $\tau \in [t, t+\Delta]$ and some Δ. Therefore, if $d/dt\, |y_k(t)| = \max_{j \in J(t)} \{d/dt\, |y_j(t)|\}$ there is a Δ_1 such that $|y_k(\tau)| \geq |y_j(\tau)|$ for $j \in J(t)$ and $\tau \in [t, t+\Delta_1]$. Furthermore the continuity of $|y_i(t)|$ implies that since $|y_i(t)| < |y_j(t)| = y(t)$ for $j \in J(t)$, $i \notin J(t)$, the same inequality holds for a neighborhood of t, namely

$$|y_i(\tau)| \leq |y_j(\tau)|$$

for $\tau \in [t, t+\Delta_2]$.

For $0 \leq \Delta t \leq \min\, [\Delta_1, \Delta_2]$ the expression $D(\Delta t)$ becomes

$$D(\Delta t) = \frac{y(t+\Delta t) - y(t)}{\Delta t} = \frac{|y_k(t+\Delta t)| - |y_k(t)|}{\Delta t}$$

where $k = \arg\max_{j \in J(t)} \{d/dt\, |y_j(t)|\}$ and hence

$$\lim_{\Delta t \to 0^+} D(\Delta t) = \frac{d}{dt}\, |y_k(t)|$$

We now have following bounds for the right-hand derivative (which we denote by RHD)

$$
\begin{aligned}
RHD(y(t)) &= \max_{j\in J(t)} \frac{d}{dt} |y_j(t)| \\
&= \max_{j\in J(t)} \frac{d}{dt} y_j(t) \cdot \operatorname{sgn} y_j(t) \\
&= \max_{j\in J(t)} k_j \cdot \operatorname{sgn} y_j(t) \cdot \left[a_{jj} d_j x_j + \sum_{i\neq j} d_j \frac{a_{ji}}{d_j} x_i \right] \\
&\le \max_{j\in J(t)} k_j |y_j| \cdot \left[a_{jj} + \sum_{i\neq j} \frac{|a_{ji}|}{d_i} d_j \right] \\
&\le k \pi y(t)
\end{aligned}
$$

In the last step we used the assumption $0 \le k \le k_j(t)$.

From the inequality

$$RHD\, y(t) \le k \pi y(t)$$

we can conclude that

$$y(t) \le \exp(k\pi t) \cdot y(0)$$

This is because

$$RHD[\exp(-k\pi t) y(t)] = \exp(-k\pi t) RHD\, y(t) - k\pi \exp(-k\pi t) y(t)$$

and hence the inequality

$$RHD\, y(t) - k\pi y(t) \le 0$$

implies

$$\exp(-k\pi t)\, RHD\, y(t) - k\pi \exp(-k\pi t) \cdot y(t) \le 0$$

and

$$RHD[\exp(-k\pi t)\, y(t)] \le 0$$

It follows from [14] that $\exp(-k\pi t)\cdot y(t)$ is decreasing in t and thus $y(0) \geq \exp(-k\pi t)\cdot y(t)$ or $y(0)\exp(k\pi t) \geq y(t)$ as was claimed.

The above proves that $y(t) \to 0$ and so does $x(t)$. The stability of the system $\dot{x} = K(t)Ax$ has been proved. Q.E.D.

Remark. The Metzler-Hicks stability conditions of Lemma 2.8 are a special case of those in Proposition 2.13. Similar conditions appear in [4], [12], [15] but not in the context of Metzler stability.

Generalizing a step further we might consider the system matrix A in (2.12) to be time varying. What conditions on $A(t)$ will guarantee stability of p^*? It can be seen by counter-example that it is not sufficient to ask that the conditions of Proposition 2.13 hold at each time instant t, i.e. $\tilde{A}(t)$ be a stable matrix for every t. However, it is known [13], [7] in the context of the study of time varying linear systems $\dot{x} = A(t)x$, that there is a class of matrices $A(t)$ for which stability of $A(t)$ for every t implies the stability of the time varying system $\dot{x} = A(t)x$. Namely, consider the system

$$\dot{x} = A(t)x \tag{2.15}$$

where the $A(t)$ matrix is of the form

$$a_{ij}(t) = \begin{cases} -\alpha_i & \text{for } i = j \\ \alpha_{ij}\cdot e_{ij}(t) & \text{for } i \neq j \end{cases} \tag{2.16}$$

and $e_{ij}(t)$ are continuous functions such that $|e_{ij}(t)| \leq 1$. In the above references [12], [7] it is shown that (2.15) is globally stable provided the matrix

$$\bar{A} = \begin{pmatrix} -a_1 & \cdots & |a_{ij}| \\ \vdots & \ddots & \\ |a_{ji}| & \cdots & -a_n \end{pmatrix} \tag{2.17}$$

is a stable M-matrix. The stability of the systems in (2.15) has been called connective stability in [13]. It can be justified if we consider the off-diagonal terms of $A(t)$ as interconnection coefficients which can vary in an arbitrary manner due to the presence of the $e_{ij}(t)$ coefficients in (2.16): Connective stability means that the stability of (2.15) is independent of the strength of the interconnection terms.

We can combine the notion of connective stability with Metzler stability, i.e. stability of all systems in (2.12) by noting that the stability condition (2.5) guarantees the stability of all the systems in (2.12). We have thus

PROPOSITION 2.18. Let $A(t)$ be defined as in (2.16) and let $\bar{A}$ in (2.17) be a stable M-matrix. The equilibrium p^* of the system

$$\frac{d}{dt} p = K(t) A(t) (p - p^*) \tag{2.19}$$

is globally asymptotically stable for all $K(t) \in \mathcal{K}_1$ as in (2.12).

Proof. As in the proof of Proposition 2.13, the fact that $\bar{A}$ is a stable M-matrix implies the existence of a positive diagonal matrix D such that $D\bar{A}D^{-1}$ is row diagonal dominant. Again let us consider the system

$$\frac{d}{dt} x = K(t) A(t) x$$

with the understanding that the stability of the origin is equivalent to the stability of p^* in (2.19). Let now $y_i = d_i x_i$ and $y(t) = \max(|y_1|, \ldots |y_n|)$. Then there exists an index set $J(t)$ such that $J(t) = \{j \mid |y_j(t)| = y(t)\}$. As in the proof of Proposition 2.13 we can show that the right hand derivative RHD(y(t)) exists and RHD $y(t) = \max_{j \in J(t)} (d/dt)\, |y_j(t)|$, when $y(t) \neq 0$. We have, for $y(t) \neq 0$

$$
\begin{aligned}
\text{RHD } y(t) &= \max_{j \in J(t)} \operatorname{sgn} y_j(t) \frac{d}{dt} y_j(t) \\
&= \max_{j \in J(t)} k_j(t) \operatorname{sgn} y_j(t) \cdot \left[-d_j a_j x_j + \sum_{i \neq j} e_{ji} a_{ji} \frac{d_j}{d_i} d_i x_i \right] \\
&\leq \max_{j \in J(t)} k_j(t) \left[-a_j |y_j| + \sum_{i \neq j} |a_{ji}| \frac{d_j}{d_i} |y_i| \right] \\
&\leq \max_{j \in J(t)} k_j(t) |y_j| \left[-a_j + \sum_{i \neq j} |a_{ji}| \frac{d_j}{d_i} \right] \\
&\leq \pi\, k\, y < 0
\end{aligned}
$$

As in the proof of Proposition 2.13 we can conclude from RHD $y(t) \leq k\pi y(t)$ that

$$y(t) \leq \exp(k\pi t)\, y(0)$$

which shows that $y(t) \to 0$ and completes the stability proof. Q. E. D.

Remark. The proofs of connective stability in [12], [13] and [7] employ the Lyapunov functions

$$v(x) = \sum_{i=1}^{n} d_i |x_i| .$$

The stability property of the matrix $\bar{A}$ implies that the d_i can be chosen in such a way that the derivative of $v(x)$ is everywhere negative. A consistent choice of d_i is not possible in case the speeds of adjustment $k_i(t)$ are time varying. This difficulty can be avoided by using the Lyapunov function of the previous proofs.

3. Nonlinear Metzler Stability Problems

3.1 A Lyapunov Function Approach

Consider the collection of systems

$$\dot{x}(t) = K(t)f(x,t) \; ; \quad K(t) \in \mathscr{K}_1$$

The origin is assumed to be an equilibrium point, i.e. $f(0,t) = 0$ for all t. The Metzler stability problem is to establish conditions on $f(x,t)$ which guarantee that the origin is globally, asymptotically stable for all $K(t) \in \mathscr{K}_1$. A natural way to attack this problem is through Lyapunov functions. Consider a scalar function $v(x,t)$ with continuous $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial t}$ and such that

(i) $\phi_1(\|x\|) \leq v(x,t) \leq \phi_2(\|x\|)$

where $\phi_i(\|x\|)$ are nonnegative, nondecreasing functions such that $\phi_i(0) = 0$, $\lim_{\|x\|\to\infty} \phi(\|x\|) = \infty$. Assume that it can be shown that for every $K(t) \in \mathscr{K}_1$, the time derivative of $v(x,t)$ along the corresponding trajectory of (2.1) satisfies

(ii) $\frac{d}{dt} v(x(t),t) \leq -\phi_3(\|x\|)$

where $\phi_3(\|x\|)$ satisfies the same conditions as ϕ_1, ϕ_2. Then the global asymptotic stability of the origin for every $K(t) \in \mathscr{K}_1$ follows by a standard

Lyapunov stability theorem [2, pg. 458]. In addition to the problem of selecting a proper $v(x,t)$, the checking of the derivative condition (ii) is extremely cumbersome.

A simple condition for (ii) to hold would be as follows. Consider for simplicity time-invariant Lyapunov function $v(x) \in C^1(R^n)$ and time invariant f functions. The derivative in (ii) can be written as

$$\frac{dv(x)}{dt} = \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} f_i(x)\, k_i \tag{2.20}$$

Unless each term $(\partial v/\partial x_i)\, f_i(x)$ in (2.20) is nonpositive and at least one is negative, there will be a choice of k_i which will make dv/dt positive (provided there is no upper bound in the k_i). In any case, the condition

$$\begin{cases} \dfrac{\partial v}{\partial x_i} f_i(x) \le 0 & i \in [1,2,\ldots,n] \\[2ex] \dfrac{\partial v}{\partial x_j} f_j(x) < 0 \quad \text{for some} & j \in [1,2,\ldots,n] \end{cases} \tag{2.21}$$

is sufficient for the stability of (2.1), coupled with some technical conditions as in the following:

LEMMA 2.22: Consider the class of systems

$$\frac{d}{dt} x = k(t)\, f(x)\,; \qquad K(t) \in \mathcal{K}_1 \tag{2.23}$$

and assume the existence of a $C^1(R^n)$ positive definite function $v(x)$ satisfying

$$\begin{cases} \dfrac{\partial v}{\partial x_i} \; f_i(x) \le 0 & i \in [1, 2, \ldots, n] \\[2em] \dfrac{\partial v}{\partial x_j} \; f_j(x) < 0 & \text{for some} \quad j \in [1, 2, \ldots, n] \end{cases}$$

Then all the systems in (2.23) are globally asymptotically stable.

Proof. Note that

$$\frac{dv}{dt} = \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} \; f_i(x) < 0$$

which satisfies the assumptions of the global as. stab. theorems of [16].

Q. E. D.

Although Lemma 2.22 is extremely simple it has interesting consequences. Let us apply it to linear systems of the form

$$\frac{d}{dt} x = K(t) \, Ax \tag{2.24}$$

with $K(t) \in \mathscr{K}_1$ as in (2.23). A quadratic Lyapunov function $v(x) = x'Mx$, $M > 0$ would satisfy the conditions of Lemma 2.22 if, for all x

$$\frac{\partial v}{\partial x_i} \; f_i = (e_i' Mx + x'Me_i)(Ax)_i = x'M_{.i} A_{i.} \, x \le 0$$

where e_i: i-th unit vector, $M_{.i}$:i-th column, $A_{i.}$:i-th row. This can only happen if the i-th row of A is proportional to the i-th column of M. These conditions can be satisfied if A is a symmetric negative definite matrix. Then $v(x) = -x'Ax$ is a positive function and

$$\frac{dv}{dt} = -x'AKAx - x'AKAx = 2x'AK(t)Ax \leq -2kx'A'Ax$$

is negative everywhere, providing thus a direct verification of the conclusion of Lemma 2.22.

It should be noted that the conditions of Lemma 2.22 are far from necessary for the stability of all systems in (2.23): In the linear case, the only systems satisfying these conditions correspond essentially to symmetric negative definite matrices. However, all A-matrices satisfying the conditions of Proposition 2.13 are Metzler stable, and these A's are not covered by Lemma 2.22. Thus the condition $\partial v/\partial x_i f_i \leq 0$ *for all* i *can fail at some* x without implying instability. On the other hand, if $\partial v/\partial x_i f_i > 0$ for some i and all x in a neighborhood N_0 of the origin, the system can't be stable as dv/dt can be made positive on N_0. Instability follows then by standard theorems [16]. The above remarks provide a weak converse to Lemma 2.22.

3.2 The Malishewski Conditions for Metzler Stability

A class of sufficient conditions for Metzler stability of nonlinear systems were presented by Malishewski [5]. In this section we examine his conditions and show that they follow in large part from Lyapunov function considerations, similar to those that lead to Lemma 2.22. First we present a brief summary of the framework of Malishewski's work.

Malishewski introduces the notion of goal-oriented elements operating jointly. Each "element" is a scalar controller and the i-th control is a scalar c_i, $i = 1, 2, \ldots, n$. The joint application of the control

values c_i, $i = 1, 2, \ldots, n$ causes each controller to receive a "mismatch" message, which depends on the controls c_i, $i = 1, \ldots, n$ applied. The mismatch message of the i-th element is determined by a deterministic function $\delta_i(c_1, c_2, \ldots, c_n)$ which is referred to as an <u>indicator function</u>. The goal of each element is to receive a zero mismatch message and therefore the joint operation of the n elements will be successful if each element applies a control of magnitude c_i^* such that $\delta_i(c_1^*, c_2^*, \ldots c_n^*) = 0$ $i = 1, 2, \ldots, n$. Two questions arise in this setup. First, the purely algebraic question of the existence of the equilibrium values c_i^*, $i = 1, 2, \ldots, n$. Second the question of whether the rules of operation of the controllers will lead to convergence of the controls to the equilibrium. Concerning the rules of operation, each element is assumed to know nothing about the functional form of the indicator functions and thus the possibility of explicit calculation of c^* by the controllers is excluded. The control strategies can be based on the knowledge of the controller's mismatch value only. Malishewski examines two kinds of control strategies. First the strategy

$$\dot{c}_i = \delta_i(c_1, \ldots, c_n) \qquad i = 1, 2, \ldots, n \tag{2.25}$$

under which the problem is a standard asymptotic stability problem. More interesting is the class of strategies described by

$$\operatorname{sgn} \dot{c}_i = \operatorname{sgn} \delta_i(c_1, c_2, \ldots, c_n) \qquad i = 1, 2, \ldots, n \tag{2.26}$$

where sgn( ) is the well known sign function. Obviously the stability of (2.26) is a Metzler stability problem as defined earlier in this section. Note that the trajectories in (2.26) must satisfy a nondegeneracy

condition, that prohibits the existence of a time sequence $\{t_n\}$ such that $\dot{c}_i(t_n) \to 0$ but $\delta_i(c(t_n)) \not\to 0$. The degeneracy condition guards against convergence to nonequilibrium points. In the same spirit, a lower bound was imposed on the adjustment rates of the Metzler stability problems we encountered earlier in this section.

A final remark should be made about Malishewski's model before we proceed to the convergence conditions. It might be realistic to impose hard constraints on the control magnitude of the form

$$a_i \le c_i \le b_i$$

These constraints can be incompatible with the controller actions in (2.25) and (2.26) as in the case where $\delta_i(c_1, \ldots, b_i, \ldots, c_n) > 0$, and thus the control policy leads to the violation of the magnitude constraint. This difficulty can be avoided by modifying the indicator functions as

$$\bar{\delta}_i(c_1, \ldots, c_n) = \begin{cases} \delta_i(c_1, \ldots, c_n) & \text{if } a_i < c_i < b_i \\ 0 & \text{if } c_i = a_i, \ \delta_i < 0 \\ & \phantom{\text{if }} c_i = b_i, \ \delta_i > 0 \end{cases}$$

Henceforth we will assume that such modification has been carried out on the indicator functions δ_i.

Malishewski considers continuous indicator functions which satisfy a class of rather formidable looking conditions and examines when these conditions imply Metzler or ordinary stability. We will show that the Metzler stability results can all be viewed as following from Lyapunov type considerations as in Lemma 2.22. Let us first review the conditions

imposed on the function indicators.

Let $\underline{c} = (c_1, \ldots, c_n)$ and $\underline{c} + \Delta\underline{c} = (c_1 + \Delta c_1, \ldots, c_n + \Delta c_n)$ be arbitrary control vectors and let

$$\Delta\delta_i(\underline{c}) = \delta_i(\underline{c} + \Delta\underline{c}) - \delta_i(\underline{c}) \qquad i = 1, \ldots, n$$

The indicator functions satisfy Condition I if the following hold: Let

$$I_> = \{i \mid \Delta c_i > 0\} ,$$

$$I_< = \{i \mid \Delta c_i < 0\} ,$$

$$I_= = \{i \mid \Delta c_i = 0\} .$$

For any $\underline{c}$, $\Delta\underline{c}$ we must have

$$\sum_{i \in I_>} \Delta\delta_i(\underline{c}) - \sum_{i \in I_<} \Delta\delta_i(\underline{c}) + \sum_{i \in I_=} |\Delta\delta_i(\underline{c})| < 0 \qquad (\Delta c \neq 0)$$

Condition II is satisfied if the following holds: Let $|\Delta c_k| = \max_i |\Delta c_i|$. We must have

$$\Delta\delta_k(\underline{c}) \cdot \Delta c_k < 0 \qquad (\Delta_{\underline{c}} \neq 0)$$

Condition III holds if

$$\sum_{i=1}^{n} \Delta\delta_i(\underline{c})\, \Delta c_i < 0 \qquad (\Delta\underline{c} \neq 0)$$

It should be noted that there is no implication relation among the three conditions. In fact it will be shown that they arise from essentially different considerations. Several ways of interpreting them are possible but the most

enlightening comment, which applies to all three, is that each condition guarantees that the i-th indicator function is most sensitive to the i-th control. Thus, a controller applying a large positive change in control is guaranteed a decrease in his mismatch message, regardless of the actions of the other controllers.

A more transparent representation of the three conditions is possible in case the indicator functions are continuously differentiable. By taking limits as $\Delta \underline{c} \to 0$ in each condition we see that each implies a corresponding condition in differential form, namely

Condition I_D

$$\frac{\partial \delta_i(\underline{c})}{\partial c_i} + \sum_{j \neq i} \left| \frac{\partial \delta_j(\underline{c})}{\partial c_i} \right| \leq 0 \qquad i = 1, \ldots, n$$

Condition II_D

$$\frac{\partial \delta_i(\underline{c})}{\partial c_i} + \sum_{j \neq i} \left| \frac{\partial \delta_i(\underline{c})}{\partial c_j} \right| \leq 0 \qquad i = 1, \ldots, n$$

Condition III_D

$$\text{All} \qquad \det \left[- \left(\frac{\partial \delta_i(\underline{c})}{\partial c_j} + \frac{\partial \delta_j(\underline{c})}{\partial c_i} \right) \right]_{i_1, \ldots, i_k} \geq 0 \qquad \text{(all minors)}$$

These conditions must hold at every $\underline{c}$ vector. If the differential conditions hold everywhere as strict inequalities, they can be integrated to yield the corresponding difference conditions.

The differential conditions above can be viewed as generalizations to nonlinear systems of well known stability conditions for linear systems.

Conditions I_D, II_D state that the matrix $(\partial\delta(\underline{c})/\partial\underline{c})$ is column and row dominant respectively and we can conclude through Gersgorin's theorem that the eigenvalues of $(\partial\delta(\underline{c})/\partial\underline{c})$ are in the L. H. P. for every $\underline{c}$. Condition III_D states that $(\partial\delta(\underline{c})/\partial\underline{c})$ is negative definite and thus its eigenvalues are also in the L. H. P. Furthermore I_D, II_D imply the conditions of Proposition 2. 13 which guarantee Metzler stability for linear systems. However, the above relations are of value mostly as mnemonics, as there is no fundamental relation between Gersgorin's theorem and the stability that conditions I_D, II_D entail. As hinted before the meaning of Maleshewski's conditions is best clarified through Lyapunov function type arguments that led to Lemma 2. 22. Namely, it can be easily seen that there exist positive scalar functions which are monotonically decreasing for every trajectory of the class (2. 26), provided the indicator functions satisfy Conditions I or II.

For indicator functions satisfying Condition I consider the positive scalar function

$$v(\underline{c}) = \sum_{i=1}^{n} |\delta_i(\underline{c})|$$

We will show that $v(\underline{c}(t))$ is decreasing in time for every trajectory of the class (2. 26). Since $v(\underline{c})$ is not everywhere differentiable with respect to $\underline{c}$, we cannot conclude that $v(c(t))$ is differentiable in t. We consider instead a derivate [14, p. 97] of $v(c(t))$, namely

$$D^+v(c(t)) = \overline{\lim_{\Delta t \to 0^+}} \frac{v(c(t + \Delta t)) - v(c(t))}{\Delta t}$$

We will show that $D^+ v(c(t)) \leq 0$. In fact,

$$D^+v(c(t)) = \overline{\lim_{\Delta t \to 0^+}} \sum_{i=1}^{n} \frac{|\delta_i(c(t+\Delta t))| - |\delta_i(c(t))|}{\Delta t} \tag{2.27}$$

$$= \overline{\lim_{\Delta t \to 0^+}} \sum_{i=1}^{n} \frac{\delta_i(c(t+\Delta t)) \operatorname{sgn} \delta_i(c(t+\Delta t)) - \delta_i(c(t)) \operatorname{sgn} \delta_i(c(t))}{\Delta t}$$

Let us assume that $\delta_i(c(t)) \neq 0$ for all i. Then for small enough Δt, we have

$$\operatorname{sgn} \delta_i(c(t+\Delta t)) = \operatorname{sgn} \delta_i(c(t)) \tag{2.28}$$

and

$$\begin{aligned} \operatorname{sgn} [c_i(t+\Delta t) - c_i(t)] &= \operatorname{sgn} [k_i \Delta t \cdot \delta_i(c(t))] \\ &= \operatorname{sgn} \delta_i(c(t)) \end{aligned} \tag{2.29}$$

Putting (2.27), (2.28), (2.29) together we get

$$D^+v(\underline{c}(t)) = \overline{\lim_{\Delta t \to 0^+}} \frac{1}{\Delta t} \sum_{i=1}^{n} \operatorname{sgn} [c_i(t+\Delta t) - c_i(t)] \cdot \Delta \delta_i(c(t))$$

Condition I implies that the expression of which the lim sup is taken is negative for Δt small, and thus $D^+v(c(t)) \leq 0$ for all t.

We assumed above that $\delta_i(c(t)) \neq 0$ for all i. It can be shown that this assumption is not necessary to show that $D^+v(c(t)) \leq 0$. In fact, let $\delta_j(c(t)) = 0$. Then

$$D^+v(c(t)) = \lim_{\Delta t \to 0^+} \frac{1}{\Delta t} \left(\sum_{i \neq j} \Delta \delta_i \cdot \operatorname{sgn} \Delta c_i + |\delta_j(c(t+\Delta t))| \right)$$

Note that Condition I stipulates that

$$\sum_{\Delta c_i \neq 0} \Delta\delta_i \operatorname{sgn} \Delta c_i + \sum_{\Delta c_j = 0} |\Delta\delta_j| < 0$$

and also that $c_j(t+\Delta t) - c_j(t) \simeq 0(\Delta t)$ while $c_i(t+\Delta t) - c_i(t) \simeq 0(\Delta t)$. A continuity argument can then be used to show that the complete *form of* Condition I implies that $D^+ v(c(t)) \leq 0$.

Using now a real analysis result from [14, p. 98] we conclude that $v(\underline{c}(t))$ is decreasing in t and is differentiable almost everywhere.

A similar monotonicity result can be given under *Condition II*. Let us assume that an equilibrium point $\underline{c}*$ exists. Consider then the scalar function

$$v(\underline{c}) = \max_i |c_i - c_i^*|$$

and consider the derivate of the function $v(\underline{c}(t))$ along a trajectory satisfying (2.26). Using the same arguments as in the proof of Proposition 2.13 we can conclude that the right hand derivative of $v(\underline{c}(t))$ exists everywhere and

$$\text{RHD}\,[v(\underline{c}(t))] = \max_{j \in J(t)} \frac{d}{dt} |c_j(t) - c_j^*|$$

where

$$J(t) = \left\{ j \,\middle|\, |c_j(t) - c_j^*| = \max_{i \in \{1, \ldots n\}} |c_i(t) - c_i^*| \right\}$$

Note that for $v(\underline{c}(t)) \neq 0$

$$\begin{aligned} \text{RHD}\,[v(\underline{c}(t))] &= \max_{j \in J(t)} \frac{d}{dt} |c_j(t) - c_j^*| \\ &= \max_{j \in J(t)} \operatorname{sgn}(c_j(t) - c_j^*) \frac{d}{dt} (c_j(t) - c_j^*) \\ &= \max_{j \in J(t)} \operatorname{sgn}(c_j(t) - c_j^*) (\delta_j(\underline{c}(t)) - \delta_j(\underline{c}^*)) k_j \end{aligned}$$

Condition II implies that the last expression is negative for $\underline{c}(t) \neq \underline{c}^*$ and hence $RHD[v(\underline{c}(t))] \leq 0$. From the result in [14, p. 98] quoted above we can conclude that $v(\underline{c}(t))$ is decreasing in t, and differentiable almost everywhere.

The scalar function $v(\underline{c}) = \max_i |c_i - c_i^*|$ is acceptable as a Lyapunov function and using standard Lyapunov type theorems it follows that

PROPOSITION 2.30 (Malishewski). Suppose that an equilibrium point $\underline{c}^*$ exists and Condition II is satisfied. There the equilibrium point $\underline{c}^*$ is globally asymptotically stable for any system in (2.26).

In contrast to $v(\underline{c}) = \max_i |c_i - c_i^*|$, the scalar function $v(\underline{c}) = \sum_{i=1}^n |\delta_i(\underline{c})|$ cannot be used to conclude stability unless it can be shown to satisfy extra conditions: The strongest result for systems in (2.26) is a convergence result for bounded trajectories.

PROPOSITION 2.31 (Malishewski). Let Condition I hold. Then each bounded trajectory of the systems in (2.26) converges to the equilibrium point.

In the monotonicity proofs above, the hypotheses i.e., Conditions I, II, were not used in their full strength. In the Condition II proof, it was only required that the inequality $\Delta c_k \cdot \Delta\delta_k < 0$ hold at $\underline{c}^*$ only, while Condition II requires the inequality to hold for every $\underline{c}$. In the Condition I convergence proofs it is only necessary that the relevant inequality hold locally, i.e. for $\underline{c}$, $\underline{c} + \Delta\underline{c}$ with $\Delta\underline{c}$ in some neighborhood of 0, while Condition I requires the inequality to hold for all $\underline{c}$ and $\Delta\underline{c} \in R^n$. It turns out that this strengthened form of I, II guarantees an important property of these models, i.e. existence and uniqueness of $\underline{c}^*$, such that $\underline{\delta}(\underline{c}^*) = 0$,

thus providing a justification for alluding to $\underline{c}*$ as the equilibrium point in *Propositions* 2.30 *and* 2.31. *Furthermore, the strong form of* Condition I guarantees that $c*$ is the only point where $dv/dt = 0$, a fact that is important if special type stability theorems are to be used which contain statements about dynamic trajectories converging to the set

$$C = \left\{ c \,\middle|\, \frac{dv(c)}{dt} = 0 \right\} .$$

The stability results that Malishewski obtains under Condition III concern the uniqueness and existence of an equilibrium point $c*$ as well as the stability of trajectories satisfying (2.25), i.e. the ordinary differential equations $c_i = \delta_i(\underline{c})$ $i = 1, \ldots, n$. No results about trajectories satisfying (2.26) i.e., $\operatorname{sgn} \dot{c}_i = \operatorname{sgn} \delta_i(\underline{c})$, appear. This unsymmetry can be clarified in terms of Lyapunov function arguments. The natural Lyapunov functions for systems satisfying condition III are

$$v_1(\underline{c}) = \sum_{i=1}^{n} \delta_i^2(\underline{c})$$

$$v_2(\underline{c}) = \sum_{i=1}^{n} (c_i - c_i^*)^2$$

and it can be easily seen that both are monotonically decreasing along trajectories satisfying (2.25). However, when dealing with Metzler-type stability as in (2.26) one must look at stronger conditions like the conditions of Lemma 2.22, namely

$$\frac{\partial v}{\partial c_i} \cdot \delta_i(\underline{c}) \leq 0$$

Computing the above expressions for v_1, v_2 and assuming differentiable

indicator functions we get

$$\frac{\partial v_1}{\partial c_i}\,\delta_i(\underline{c}) = 2\sum_{j=1}^{n}\delta_j(\underline{c})\,\frac{\partial \delta_j}{\partial c_i}\cdot\delta_i(\underline{c}) \le 0 \qquad (2.32)$$

$$\frac{\partial v_2}{\partial c_i}\,\delta_i(c) = 2\,(c_i - c_i^*)\,\delta_i(c) \qquad \le 0 \qquad (2.33)$$

Neither inequality is implied by Condition III. However, when trajectories of (2.25) are considered the total derivatives become

$$\frac{dv_1(\underline{c})}{dt} = 2\sum_{j,i}\delta_i\,\delta_j\,\frac{\partial \delta_j}{\partial c_i} \le 0$$

$$\frac{dv_2(\underline{c})}{dt} = 2\sum_{i=1}^{n}(c_i - c_i^*)\,(\delta_i(c) - \delta_i(c^*)) \le 0$$

if Conditions III_D, III hold. Thus Lyapunov theorems based on v_1, v_2 can be proven for (2.25) type trajectories but not those in (2.26).

We saw above that Conditions I, II entail very desirable stability properties, much more desirable than those of Condition III. However, there is a rather important drawback in their formulations: They are not invariant with respect even to a change of scale: In an economic, say, model let us assume that either Condition I or II is satisfied. Consider now the possibility of another observer expressing the same situation but in terms of the variables

$$c_i^n(t) = \mu_i\, c_i(t) \qquad i = 1, 2, \ldots, n$$

with μ_i positive time invariant scalars. The model in terms of the c_i^n will have the same stability properties as the one expressed in terms of c_i, but it will not necessarily satisfy neither condition I nor II. This can be seen most easily by examining the differential conditions and observing that diagonal dominance properties are not necessarily preserved under the operation $K(\cdot)K^{-1}$ where K is a positive diagonal matrix. On the other hand Condition III is preserved under such transformations. The conclusion is that if a certain economic model involving indicator function has been constructed and one wants to test it for Metzler type stability, testing for Conditions I, II will most probably give negative results even though the model "essentially" satisfies those conditions. The reason is that one can never be sure of choosing the "right" scale and most likely he won't.

A simple way to get around this difficulty is to modify Conditions I, II, as follows: Condition $I_M(II_M)$ is satisfied if there exists a positive scale transformation which transforms the indicator functions into ones for which Condition I(II) is satisfied. Unfortunately there doesn't seem to exist an effective way of verifying these modified conditions which are, therefore, of little value. It is interesting to note though that if $\partial\underline{\delta}/\partial\underline{c}$ is diagonally dominant, the matrix $K\,(\widetilde{\partial\underline{\delta}/\partial\underline{c}})\,K^{-1}$ will be a stable M-matrix, for every k, where

$$\left\{\widetilde{\frac{\partial\underline{\delta}}{\partial\underline{c}}}\right\} = \begin{bmatrix} \frac{\partial\delta_i}{\partial c_i} & & \left|\frac{\partial\delta_i}{\partial c_j}\right| \\ & \ddots & \\ \left|\frac{\partial\delta_j}{\partial c_i}\right| & & \ddots \end{bmatrix}$$

The natural conjecture that Metzler stability is implied by the condition that $\{\partial\underline{\delta}/\delta\,\underline{c}\}$ is a stable M-matrix can be easily shown to be false. What is true is the following generalization of the Metzler-Hicks conditions to nonlinear systems

PROPOSITION 2.34 (Malishewski). Suppose that $\partial\underline{\delta}(\underline{c})/\partial\underline{c}$ is a stable M-matrix for all $\underline{c}$ and suppose that there exists vectors $\underline{c}^1 < \underline{c}^2$ such that $\delta_i(c^1) \geq 0$, $\delta_i(c^2) \leq 0$, $i = 1, 2, \ldots, n$. Let $S = \{c \mid c^1 \leq c \leq c^2\}$. Then there exists a unique equilibrium c^* in S and each trajectory satisfying (2.26) that starts in S, i.e. $c(0) \in S$, converges to c^*.

The proof of this proposition relies heavily in arguments involving "reflecting barrier boxes." It is not directly related to Lyapunov function type proofs, and can't be easily explained by the arguments used to explain Conditions I, II. The hypotheses of Proposition 2.34 are interesting in that they both generalize the Hicks conditions and are invariant under scale transformation. In that sense they are a satisfactory generalization of the results that were obtained by Metzler in the linear system case.

4. The Hicks Conditions in the Analysis of Large Systems

The goal of large scale system theory is to identify systems of high dimensionality, the properties of which can be efficiently determined by special methods of the theory. For example, the decomposition theory of linear programming identifies a class of linear programming problems which can be efficiently solved by the Dantzig-Wolfe Decomposition

Algorithm. The Hicks conditions play an interesting role in large scale system theory by establishing a class of systems about which one can efficiently answer stability questions [2], [12], [7], [15]. We will survey some typical results and we will provide extensions to Metzler stability.

Suppose we are interested in the stability of the high dimensional system

$$\dot{x} = f(x, t) \tag{2.35}$$

Assume that the system possesses unique solutions for every initial condition and that the origin is the unique equilibrium point. The following structure is postulated on system (2.35): It is "composed" of S interconnected dynamic subsystems; the i-th of which has a state $x_i \in R^{n_i}$. The "total" system state is $x = (x_1, \ldots, x_S)$. The system equation (2.35) is expressible as

$$\dot{x}_i = g_i(x_i, t) + h_i(t, x) \qquad i = 1, \ldots, S \tag{2.36}$$

System (2.36) can be conceived as S systems of the form

$$\dot{x}_i = g_i(x_i, t) \tag{2.37}$$

additively interconnected through the S functions $h_i(t, x)$, $i = 1, \ldots, S$.

Each unforced subsystem (2.37) is assumed to be globally asymptotically stable and to possess a scalar Lyapunov function from which its stability properties are evident. Namely it is assumed that there exist functions $v_i(x_i, t)$ such that

$$\phi_{i1}(\|x_i\|) \leq v_i(t, x_i) \leq \phi_{i2}(\|x_i\|)$$

$$\dot{v}_i(t, x_i) = \frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_i} \cdot g_i \le - \phi_{i3}(\|x_i\|)$$

where $\phi_{ij}(e)$ are increasing, C^1 functions from R^+ to R^+ and such that $\phi_{ij}(0) = 0$, $\phi_{ij}(\infty) = \infty$.

In the search for a Lyapunov function for the entire system (2.36), an intuitively appealing idea is to construct one on the basis of the v_i. In general this is not possible. A case in which it <u>is</u> possible though can be described as follows: The crucial assumption is that the term $\partial v_i/\partial x \cdot h_i$ can be bounded by the functions $\phi_{i3}(\|x\|)$. In particular, assume that

$$\frac{\partial v_i}{\partial x} \cdot h_i(x, t) \le \sum_{j=1}^{S} \xi_{ij}(x, t)\, \phi_{j3}(\|x_j\|) \tag{2.38}$$

and that the ξ_{ij} are bounded

$$\sup_{x,t} |\xi_{ij}(x, t)| = \theta_{ij} \le \infty \quad . \tag{2.39}$$

With the above assumptions, the following bound on dv_i/dt is immediate

$$\begin{aligned}
\frac{dv_i}{dt} &= \frac{dv_i}{dt} + \frac{\partial v_i}{dx}\,(g_i(x_i, t) + h_i(x, t)) \\
&\le - \phi_{i3}(\|x_i\|) + \sum_{j=1}^{S} \xi_{ij}(x, t)\, \phi_{j3}(\|x_j\|) \\
&\le - \phi_{i3}(\|x_i\|) + \sum_{j=1}^{S} \theta_{ij}\, \phi_{j3}(\|x_j\|)
\end{aligned} \tag{2.40}$$

We can rewrite (2.40) in vector form

$$\dot{v} = (\dot{v}_1, \ldots, \dot{v}_s)^T \le A(\phi_{13}(\|x_1\|), \ldots, \phi_{s3}(\|x_s\|))^T \qquad (2.41)$$

where A is the M-matrix $\{a_{ij}\} = \{-\delta_{ij} + \theta_{ij}\}$ where δ_{ij} is the Kronecker δ and θ_{ij} is defined in (2.39). Clearly $\{a_{ij}\}$ is an M-matrix and its stability is equivalent to the Hicks conditions. The stability of $\{a_{ij}\}$ is related to the stability of (2.36) by the following

PROPOSITION 2.42. The system (2.36) is globally asymptotically stable if $A = \{a_{ij}\}$ satisfies the Hicks conditions.

Proof. If A satisfies the Hicks conditions, there exists a positive vector $\underline{d}$ such that $\underline{d}'A$ is a negative vector [4]. Consider the scalar function

$$V(t,x) = \underline{d}' \cdot \underline{v} = \sum_{i=1}^{S} d_i \, v_i(t, x_i) \quad .$$

Then from (2.41) we get

$$d'\dot{\underline{v}} = \dot{V} \le \underline{d}A(\phi_{13}, \phi_{23}, \ldots, \phi_{S3})^T < 0$$

and stability follows from Lyapunov type arguments.

The stability analysis of a $n = \sum_{i=1}^{S} n_i$ - dimensional problem reduces, under the above conditions to checking the Hicks conditions for a $S \times S$ matrix. This simplification can be applied only to systems satisfying (2.38), (2.39), which are apparently quite restrictive. Still, there are reports of applications of this analysis to electric power problems [9], which show that the class of systems examined above is restricted but interesting for applications.

Under slightly stronger assumptions than those of Proposition 2.42, a Metzler stability result can be obtained for systems of the form (2.36).

Let us assume that each subsystem in (2.36) has its particular rate of adjustment, i.e.,

$$\dot{x}_i = K_i \quad [g_i(x_i, t) + h_i(x, t)] \qquad i = 1, \ldots, S \tag{2.43}$$

where $K_i = k_i(t) \cdot I$, $k_i(t)$ continuous, positive function of t and I a $n_i \times n_i$ identity matrix. We assume as before that there is a lower bound $0 < \underline{k}_i \le k_i(t)$ for $i = 1, \ldots, S$.

Let us assume now that the subsystems possess norm-bounded Lyapunov functions $v_i(x_i)$ such that

$$\begin{aligned} &\eta_{i1} \|x_i\| \le v_i(x_i) \le \eta_{i2} \cdot \|x_i\| \\ &\frac{\partial v_i}{\partial x_i} \cdot g_i(x_i, t) \le -\eta_{i3} \cdot \|x_i\| \end{aligned} \tag{2.44}$$

and an interconnection bound

$$\frac{\partial v_i}{\partial x_i} \cdot h_i(x_i, t) \le \sum_{j=1}^{S} \xi_{ij}(x, t) \|x_j\| \tag{2.45}$$

and

$$\sup_{x,t} |\xi_{ij}(x, t)| \le \theta_{ij}$$

holds.

Then the following bound holds for dv_i/dt

$$\begin{aligned} \frac{dv_i}{dt} &= k_i(t) \cdot \frac{\partial v_i}{\partial x_i} (g_i + h_i) \le - k_i \, \eta_{i3} \|x_i\| + \sum_{j=1}^{s} k_i \, \theta_{ij} \|x_j\| \\ &\le - k_i \left[\frac{\eta_{i3}}{\eta_{i2}} \, v_i - \sum_{j=1}^{s} \frac{\theta_{ij}}{\eta_{j1}} \, v_j \right] \end{aligned} \tag{2.46}$$

Writing (2.12) in matrix form we get

$$\left(\frac{dv_i}{dt}, \ldots, \frac{dv_s}{dt}\right)^T \leq K \cdot A(v_1, \ldots, v_s)^T \tag{2.47}$$

where

$$K = \begin{pmatrix} k_1 & & \\ & \ddots & \\ & & k_s \end{pmatrix}$$

$$A = \{a_{ij}\} = \{ -\delta_{ij} \eta_{i3} \eta_{i2}^{-1} + \theta_{ij} \eta_{j1}^{-1} \} \tag{2.48}$$

We now have

PROPOSITION 2.49. *Under assumptions* (2.44), (2.45) *all the systems described by* (2.43) *are stable provided* A *is a stable matrix.*

<u>Proof</u>. A is a stable M-matrix. Hence there exists a positive diagonal matrix $D = \text{diag}\{d_1, \ldots, d_s\}$ such that DAD^{-1} satisfies, according to the result already quoted in Proposition 2.13 from [8], the inequalities

$$a_{ii} + \sum_{j \neq i} a_{ij} \frac{d_i}{d_j} \leq -\pi < 0 \tag{2.50}$$

Let now

$$v(x) = \max_{i \in \{1, \ldots, s\}} \{d_i v_i(x)\}$$

and

$$J(t) = \{j \mid d_j v_j(x(t)) = v(x(t))\}$$

Along a trajectory of any system in (2.43) the $v_i(x(t))$ are continuously differentiable functions of t. Using the same arguments that were used in the proof of Proposition 2.13 we can show that the right hand derivative of $v(x(t))$ exists everywhere and

$$\begin{aligned} \text{RHD}\,[v(x(t))] &= \max_{j\in J(t)} d_j \frac{d}{dt} v_j(x(t)) \\ &\le \max_{j\in J(t)} k_j \left[a_{jj}(d_j v_j) + \sum_{i\neq j} a_{ji} \frac{d_j}{d_i} (d_i v_i) \right] \\ &\le -k \cdot \pi \cdot v(\underline{x}(t)) \end{aligned}$$

where $k = \min\{k_j(t)\}$. As in Proposition 2.13 we can conclude that $v(\underline{x}(t)) \le \exp(-k\pi t) v(\underline{x}(0))$ and hence $v(t) \to 0$, $\underline{x}(t) \to 0$. Q. E. D.

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III. THE APPLICATION OF HJB-INEQUALITIES TO OPTIMIZATION

1. Introduction

In this Section we examine some issues in large scale optimization. We consider a collection of S control systems

$$\dot{x} = f_i(x_i, u_i, t) \; ; \qquad x_i(0) = x_{i0} \tag{3.1}$$

where $x_i \in R^{n_i}$ for $i = 1, \ldots, S$ and the cost functional

$$J(x_0, u) = \sum_{i=1}^{S} J_i(x_i, u_i) = \sum_{i=1}^{S} \int_{t_0}^{T} \left[L_i(x_i, u_i; t) \; dt + \phi_i(x_{iT}) \right] \tag{3.2}$$

It can be seen that finding the infimum of (3.2) over the trajectories of (3.2) decouples to S lower dimensional problems, namely

$$\begin{aligned} &\inf_{u_i} \int_{t_0}^{T} L_i(x_i, u_i, t) \; dt + \phi_i(x_{iT}) \\ &\dot{x}_i = f_i(x_i, u_i, t) \qquad x_i(0) = x_{i0} \end{aligned} \tag{3.3}$$

The decomposition of problem (3.2) will be invalid the moment any coupling is introduced among the subsystems (3.1). Let $x = (x_1, \ldots, x_S) \in R^n$ be the total state vector and $g_i(x,t) : R^n \times [t_0, T] \to R^{n_i}$ be a continuous function. Then the coupling between the subsystems can be modelled additively as

$$\dot{x}_i = f_i(x_i, u_i; t) + g_i(x, t) \tag{3.4}$$

Naturally, (3.4) is not the most general form of coupling between subsystems.

We are interested in comparing the optimization problems corresponding to the decoupled system (3.1) and the coupled one (3.4). More generally, we will consider the optimization of the functional

$$J(x_0, u; t_0) = \int_{t_0}^{T} L(x, u; t) \; dt + \phi(x_T) \tag{3.5}$$

over the trajectories of the systems

$$\dot{x} = f(x, u; t) \qquad x(t_0) = x_0 \tag{3.6}$$

and

$$\dot{x} = f(x, u; t) + g(x; t) \qquad x(t_0) = x_0 \tag{3.7}$$

where $x \in R^n$, $u \in R^m$. Two questions become of interest in this context. First, is it possible to compare the payoffs corresponding to systems (3.6) and (3.7) without explicitly solving for the optimum of the, presumably, complicated system (3.7)? Second, how good are "naive" optimization policies for (3.7)? If, say, the policy $u(x, t)$ that has been proven optimal for (3.6) is applied to (3.7) how good is it going to be? We provide conditions under which it is possible to answer such questions.

The techniques used in this Section is that relating to inequalities of the Hamilton Jacobi Bellman type (henceforth abbreviated as HJB) inequalities. J. C. Willems [8, 9] has used similar inequalities, which he calls dissipation inequalities, in stability theory. We hope to show here that they are as interesting in optimization theory. These inequalities play an important role in a class of iterative optimization techniques introduced

by Bellman, a special case of which is Kleinman's algorithm [5] for the solution of the algebraic Riccati equation. In the last section of this chapter we extend the applicability of these algorithms through the use of HJB inequalities. The extended form of the algorithm will be needed in Section 4.

2. Hamilton Jacobi Bellman (HJB) Inequalities

Consider first a finite time optimization problem, namely

$$\operatorname*{Inf}_{u} \int_{t_0}^{T} L(x, u; t)\ dt + \phi(x_T) \tag{3.8}$$

subject to dynamics and initial condition

$$\dot{x} = f(x, u; t) \qquad x(t_0) = x_0 \tag{3.9}$$

and the control constraint $u(\cdot) \in \mathscr{U}$

$$\mathscr{U} = \left\{ u: R^n \times [t_0, T] \rightarrow \Omega \subset R^m \text{ and piecewise continuous} \right\} \tag{3.10}$$

Let $V(x, t)$ be the infimum value of the objective function over the interval $[t, T]$ starting at state x at time t. The following theorem states that $V(x, t)$ can be characterized through a partial differential equation, the Hamilton-Jacobi-Bellman equation.

THEOREM 3.11 [7, p. 192] Suppose there exists a differentiable function $V : R^n \times [t_0, T] \rightarrow R$ satisfying

$$\frac{\partial V}{\partial t} + \text{Min}\left\{ L(x,u,t) + \frac{\partial V}{\partial x} \cdot f(x,u;t) \,\middle|\, u \in \Omega \right\} = 0 \qquad (3.12)$$

and the boundary condition

$$V(x,T) = \phi(x)$$

Suppose there exists a function $u : R^n \times [t_0, T] \to \Omega$ piecewise continuous in t and Lipschitz in x satisfying

$$L(x,u;t) + \frac{\partial V}{\partial x} f(x,u;t) = \text{Min}\left\{ L(x,u;t) + \frac{\partial V}{\partial x} f(x,u;t) \,\middle|\, u \in \Omega \right\}$$

Then u is an optimal feedback control for the problem (3.8) and V is the optimal value function.

It is of interest to consider the inequalities corresponding to (3.12). We do so in the following

PROPOSITION 3.13. (i) Let us assume that the statement of Theorem 3.11 is satisfied. Suppose there exists a differentiable function $U : R^n \times [t_0; T] \to R$ satisfying

$$\frac{\partial U}{\partial t} + \text{Min}\left\{ L(x,u;t) + \frac{\partial U}{\partial x} \cdot f(x,u;t) \,\middle|\, u \in \Omega \right\} \geq 0 \qquad (3.14)$$

and the boundary condition

$$U(x;T) \leq \phi(x) \qquad (3.15)$$

Then $U(x,t) \leq V(x;t)$ for $(t,x) \in [t_0, T] \times R^n$.

(ii) As in (i) let there be a differentiable $U(x,t)$ and a $\bar{u}(x,t) \in \mathcal{U}$ such that

$$\frac{\partial U}{\partial t} + L(x, \bar{u}; t) + \frac{\partial U}{\partial x} \cdot f(x, \bar{u}; t) \leq 0 \tag{3.16}$$

and

$$U(x; T) \geq \phi(x) \tag{3.17}$$

Then $U(x,t) \geq V(x,t)$ for $(x,t) \in R^n \times [t_0, T]$.

<u>Proof</u>. Consider the value of the cost functional

$$J(x, u, t_1) = \int_{t_1}^{T} L(x, u, t)\ dt + \phi(x_T)$$

when the control law $u(x,t) \in U$ is applied to the system

$$\dot{x} = f(x, u(x,t), t)$$

with initial condition $x(t_1) = x_1$.

We now add and subtract

$$\int_{t_1}^{T} \frac{d}{dt} U(x,t)\ dt$$

(U is computed along the trajectory) to $J(x_1, u, t_1)$:

$$J(x_1, u, t_1) = \int_{t_1}^{T} L(x, u, t)\ dt + \phi(x_T) + \int_{t_1}^{T} \left[- \frac{d}{dt} U(x,t) + \frac{\partial U}{\partial t} + \frac{\partial U}{\partial x} f(x, u, t) \right] dt$$

$$= U(x_1, t_1) + \phi(x_T) - U(x_T, T) + \int_{t_1}^{T} \left[\frac{\partial U}{\partial t} + L + \frac{\partial U}{\partial x} f \right] dt \tag{3.18}$$

To prove part (i) assume that (3.14) and (3.15) hold. Now (3.14) implies that

$$\frac{\partial U}{\partial t} + L(x,u,t) + \frac{\partial U}{\partial x} f(x,u,t) \geq 0$$

for any x, u, and hence the integral

$$\int_{t_1}^{T} \left[\frac{\partial U}{\partial t} + L + \frac{\partial U}{\partial x} f \right] dt$$

is nonnegative. The difference $\phi(x_T) - U(x_T, T)$ is also nonnegative by (3.15), and thus (3.18) implies that

$$J(x_1,u,t_1) \geq U(x_1,t_1)$$

for any control law $u(x,t)$ and any x_1, t_1. Since $V(x_1,t_1) = \min_u J(x_1,u,t_1)$ it follows that $V(x_1,t_1) \geq U(x_1,t_1)$, which proves part (i).

To prove part (ii), assume that (3.16) and (3.17) hold. This time we can't conclude from (3.16) that

$$\frac{\partial U}{\partial t} + L(x,u,t) + \frac{\partial U}{\partial x} f(x,u,t) \leq 0$$

for arbitrary x, u; Equation (3.16) states that

$$\frac{\partial U}{\partial t} + L(x,\bar{u},t) + \frac{\partial U}{\partial x} f(x,\bar{u},t) \leq 0$$

for a specific u, namely

$$u(x,t) = \bar{u}(x,t) \quad .$$

To prove part (ii) we consider the value function $J(x_1, \bar{u}(x,t), t_1)$ with $\bar{u}$ defined above. We will show next that $J(x_1, \bar{u}, t_1) \le U(x_1, t_1)$ and hence $V(x_1, t_1) \le J(x, \bar{u}, t_1) \le U(x_1, t_1)$.

Looking at (3.18) with $u(x,t) = \bar{u}(x,t)$ we get that the integral

$$\int_{t_0}^{T} \left[\frac{\partial U}{\partial t} + L(x, \bar{u}, t) + \frac{\partial U}{\partial x} f(x,u,t) \right] dt$$

is nonpositive and so is $\phi(x_T) - U(x_T, T)$. It follows from (3.18) that $J(x_1, \bar{u}, t_1) \le U(x_1, t_1)$. Q.E.D.

Remark. In the above Proposition we can replace the min operator by any of max, inf, sup operators without any significant change in the results.

For the class of infinite duration optimization problems, similar results have been derived by Willems [9]. In particular, consider the following optimization problems

$$V_f(x_0) = \inf_u \int_{t_0}^{\infty} L(x,u) \; dt$$

$$V(x_0) = \inf_u \int_{t_0}^{\infty} L(x,u) \; dt; \quad \lim_{t \to \infty} x_t = 0$$

$$V_n(x_0) = \inf_{u,T} \int_{t_0}^{T} L(x,u) \; dt$$

all subject to the dynamic constraints

$$\dot{x} = f(x,u) \qquad x(t_0) = x_0 \; ; \; u \in U \; .$$

Clearly $V_n(x_0) \leq V_f(x_0) \leq V(x_0)$ for all x_0. We have the following proposition relating V, V_f, V_n to the solutions of a time invariant HJB inequality.

PROPOSITION 3.20. Consider the HJB inequalities

$$\inf \left[L(x,u) + \frac{\partial U}{\partial x} f(x,u) \,\middle|\, u \in \Omega \right] \geq 0 \tag{3.21}$$

$$\inf \left[L(x,\bar{u}) + \frac{\partial V}{\partial x} f(x,\bar{u}) \,\middle|\, u \in \Omega \right] \leq 0 \tag{3.22}$$

and suppose that there exists a nonpositive function $V_1(x)$ satisfying (3.21) a nonnegative function $V_2(x)$ and a $\bar{u}(x,t)$ satisfying (3.22). Then

$$V_1(x) \leq V_n(x) \leq V_f(x) \leq V_2(x) \tag{3.23}$$

Furthermore, if $V_3(x)$ satisfies (3.21) and $V_3(0) = 0$, we have

$$V_3(x) \leq V(x)$$

<u>Proof</u>. Consider the integral $\int_{t_0}^{T} L(x,u)\,dt$ subject to (3.19). Adding and subtracting to it

$$\int_{t_0}^{T_0} V_i(x)\, dt \qquad (i = 1, 2, 3)$$

we get

$$J(x_0; T) = \int_0^T \left[L(x,u) - \dot{V}_i + \frac{\partial V_i}{\partial x} f(x,u) \right] dt$$

$$= V_i(x_0) - V_i(x_T) + \int_0^T \left[L(x,u) + \frac{\partial V_i}{\partial x} \cdot f(x,u) \right] dt \tag{3.24}$$

For $i = 1$, equations (3.21) and (3.24) imply that $V_1(x_0) \le J(x_0, T)$ for any x_0, any T and any control u, hence $V_1(x_0) \le V_n(x_0)$. Similarly for $i = 2$ we get $V_2(x_0) \ge V_f(x_0)$ by considering a control law that satisfies (3.22). Finally for $i = 3$ and $T \to \infty$ we get from (3.24) that

$$\inf_u \left\{ \lim_{\substack{T\to\infty \\ x(T)\to\infty}} J(x_0; T) \right\} = V(x_0) \ge V_3(x_0)$$

Q. E. D.

Remark. This proposition is due to Willems [9] except for the parts relating to (3.22).

3. Comparison of Systems

3.1 General Results

We now consider our original problem, to compare the infima of the functional

$$J(x_0, u; t_0) = \int_{t_0}^{T} L(x, u; t)\, dt + \phi(x_T) \tag{3.1}$$

over the trajectories of the systems

$$\dot{x} = f(x, u; t) \qquad x(t_0) = x_0 \tag{3.2}$$

and

$$\dot{x} = f(x, u; t) + g(x, t) \quad x(t_0) = x_0 \tag{3.3}$$

Let us assume that the value functions corresponding to these problems are $V_1(x; t)$, $V_2(x; t)$. We have the following comparison-type proposition

PROPOSITION 3.25: Suppose that $V_1(x;t)$, $V_2(x;t)$ are continuously differentiable, satisfy the corresponding Hamilton-Jacobi-Bellman equations

$$\frac{\partial V_1}{\partial t} + \min\left\{ L(x,u;t) + \frac{\partial V_1}{\partial x} \cdot f(x,u;t) \,|\, u \in \Omega \right\} = 0 \tag{3.26}$$

and

$$\frac{\partial V_2}{\partial t} + \min\left\{ L(x,u;t) + \frac{\partial V_2}{\partial x} \cdot [f(x,u;t) + g(x;t)] \,|\, u \in \Omega \right\} = 0 \tag{3.27}$$

and the common boundary condition $V(x;T) = \phi(x)$. If

$$\frac{\partial V_1}{\partial x} \cdot g(x;t) \le 0$$

for all $x \in R^n$, $t \in [t_0, T]$, where $g(x;t)$ is the "modification term" in (3.3) we have $V_1(x;t) \ge V_2(x;t)$. If

$$\frac{\partial V_1}{\partial x} \cdot g(x,t) \ge 0$$

then $V_1(x;t) \le V_2(x;t)$.

<u>Proof.</u> Substituting $V_1(x;t)$ in (3.27) we get

$$\frac{\partial V_1}{\partial t} + \min\left\{ L(x,u;t) + \frac{\partial V_1}{\partial x}\, f(x,u;t) \,|\, u \in \Omega \right\} + \frac{\partial V_1}{\partial x} \cdot g(x,t)$$

$$= \frac{\partial V_1}{\partial x} \cdot g(x,t) \le 0$$

Hence $V_1(x,t)$ *satisfies* a HJB *inequality* and by Proposition 3.13, $V_1(x;t) \ge V_2(x;t)$. The second statement of the proposition follows in the same manner. Q.E.D.

Similar comparison results can be derived from Proposition 3.20.

Let

$$\begin{aligned} V^1(x_0) &= \inf_u \int_{t_0}^{\infty} L(x,u)\, dt; \quad \lim_{t\to\infty} x(t) = 0 \\ \dot{x} &= f(x,u) \qquad x(t_0) = x_0 \end{aligned} \tag{3.28}$$

and

$$\begin{aligned} V^2(x_0) &= \inf_u \int_{t_0}^{\infty} L(x,u)\, dt; \quad \lim_{t\to\infty} x(t) = 0 \\ \dot{x} &= f(x,u) + g(x) \end{aligned} \tag{3.29}$$

Assume furthermore that V^1, V^2 are continuously differentiable and satisfy the HJB equations

$$\min \left\{ L(x,u) + \frac{\partial V^1}{\partial x} f(x,u) \,\middle|\, u \in \Omega \right\} = 0$$

$$\min \left\{ L(x,u) + \frac{\partial V^2}{\partial x} [f(x,u) + g(x)] \,\middle|\, u \in \Omega \right\} = 0$$

as well as the boundary condition $V^1(0) = V^2(0) = 0$. We then have

PROPOSITION 3.30: The inequality

$$\frac{\partial V^1}{\partial x} g(x) \geq 0 \qquad \forall x$$

implies that $V^1(x) \leq V^2(x)$.

<u>Proof.</u> Substitute $V^1(x)$ in the HJB equation satisfied by $V^2(x)$ and note

$$\inf\left\{L(x,u) + \frac{\partial V^1}{\partial x}[f(x,u)+g(x)] \,\middle|\, u \in \Omega\right\}$$

$$= \inf\left\{L(x,u) + \frac{\partial V^1}{\partial x} f(x,u) \,\middle|\, u \in \Omega\right\} + \frac{\partial V^1}{\partial x} g(x)$$

$$= \frac{\partial V^1}{\partial x} g(x) \geq 0$$

It follows from Proposition 3.20 that $V^1(x) \leq V^2(x)$. Q.E.D.

3.2 Linear Quadratic Problems

It is well known [4] that for the linear quadratic minimization problem

$$\min_u \int_{t_0}^{T} x'C'Cx + u'u \quad dt \tag{3.31}$$

$$\dot{x} = Ax + Bu \qquad x(t_0) = x_0$$

with (A, B, C) a minimal realization, the optimal value function is $V_1(x_0; t_0) = x_0' K_1(t_0; T, 0) x_0$ where $K_1(t; T, 0)$ is the solution of the Riccati equation

$$\dot{K}_1 + A'K_1 + K_1 A + C'C = K_1 BB' K_1 \; ; \quad K_1(T; T, 0) = 0 \tag{3.32}$$

Furthermore $\lim_{T\to\infty} K_1(t; T, 0) = K_1^+$, the unique stabilizing solution of the Algebraic Riccati equation (ARE)

$$A'K_1^+ + K_1^+ A + C'C = K_1^+ BB' K_1^+ \tag{3.33}$$

Similar results hold when the system matrix A is replaced by $A + H$. We

denote the value function as $V_2(x_0; t_0) = x'_0 K_2(t_0; T, 0) x_0$ and the steady state value matrix by $\lim_{T\to\infty} K_2(t; T, 0) = K_2^+$.

Using Proposition 3.25 we get that the condition $x'K_1(t; T, 0)Hx \geq 0$ (≤ 0) or equivalently

$$K_1(t; T, 0)H + H'K_1(t; T, 0) \geq 0 \qquad (\leq 0)$$

implies that $K_2(t; T, 0) \geq K_1(t; T, 0)$ $(\leq)$.

Proposition 3.20 implies a similar result for the steady state values, namely

$$K_1^+ H + H'K_1^+ \geq 0 \quad \Rightarrow \quad K_1^+ \leq K_2^+$$

3.3 Bounds for Suboptimal Controllers

We now consider the problem of characterizing the performance of suboptimal controllers for interconnected systems that was posed in the introduction. Bailey and Laub [1] consider the following situation: Assume that for the optimization problem

$$\begin{aligned} &\max_u \int_{t_0}^{\infty} L(x, u; t)\, dt \; ; \qquad L(x, u; t) \geq 0 \\ &\dot{x} = f(x, u; t) \; ; \qquad x(t_0) = x_0 \end{aligned} \tag{3.34}$$

there exists an optimal value function $V_1(x; t)$ and a piecewise continuous control $u_1(x; t)$ satisfying the relation

$$\frac{\partial V_1}{\partial t} + \max \left[L(x, u; t) + \frac{\partial V_1}{\partial x} f(x, u; t) \,\middle|\, u \in \Omega \right] = 0 \tag{3.35}$$

the boundary condition $V(0; t) = 0$ and

$$u_1(x,t) = \arg\max \left[L(x,u;t) + \frac{\partial V_1}{\partial x} f(x,u,t) \,\middle|\, u \in \Omega \right]$$

Assume furthermore that $f(0,u;t) = 0$; so $x = 0$ is an equilibrium point of (3.34). Bailey and Laub consider what will happen if the control $u_1(x,t)$ is applied to the modified system

$$\dot{x} = f(x,u;t) + g(x,u;t) \qquad x(t_0) = x_0 \tag{3.36}$$

(with the assumption $g(0,u;t) = 0$, so that $x = 0$ is an equilibrium of (3.36)). They show that if the condition

$$L(x,u_1;t) - \frac{\partial V_1}{\partial x} \cdot g(x,u_1;t) \geq 0 \tag{3.37}$$

holds for all x, t then

i) The suboptimal control $u_1(x,t)$ when applied to (3.36) results in a trajectory that converges to 0 as $t \to \infty$.

ii) If $V_S(x_0;t_0)$ is the payoff that results when the control law u_1 is applied to (3.36) then

$$V_S(x;t) \leq q\, V_1(x;t)$$

where

$$q = \sup_{\tau \geq t} \frac{L(x,u_1;\tau)}{L(x,u_1;\tau) - \dfrac{\partial V_1}{\partial x} g(x,u_1;\tau)} < \infty$$

The above upper bound is of interest in characterizing the performance of the suboptimal controller. Bailey and Laub point out the need for a lower

bound on its performance. Conditions for such a bound can be easily derived using arguments of this section.

Let us write $V_S(x;t)$ as

$$V_S(x_0;t_0) = \int_{t_0}^{\infty} L(x,u_1;t)\ dt + \int_{t_0}^{\infty} -\frac{d}{dt} V_1 + \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x}\ [f+g]\ dt$$

$$= \lim_{T\to\infty} \left\{ V_1(x_0;t_0) - V_1(x_T;T) + \int_{t_0}^{T} \left[V_{1t} + L(x,u_1;t) + \frac{\partial V_1}{\partial x} f \right] + \frac{\partial V_1}{\partial x} g\ dt \right\}$$

$$= V_1(x_0;t_0) + \int_{t_0}^{\infty} \frac{\partial V_1}{\partial x}\ g(x,u_1;t_0)\ dt\ .$$

The last equality follows from the fact that $\lim_{T\to\infty} x_T = 0$, V_1 is continuous and that V_1 satisfies a HJB equation. Therefore, if we require in addition to the Bailey-Laub condition

$$L(x,u_1;t) \geq \frac{\partial V_1}{\partial x}\ g(x,u_1;t) \tag{3.37}$$

that

$$L(x,u_1;t) \geq \frac{\partial V_1}{\partial x}\ g(x,u_1;t) \geq 0 \tag{3.38}$$

we obtain an upper as well as a lower bound on the performance of the suboptimal controller u_1, namely

$$q\ V_1(x,t) \geq V_S(x,t) \geq V_1(x,t) \tag{3.39}$$

4. Optimization Algorithms

4.1 Bellman's Algorithm

In 1954 Bellman [3] proposed a monotone policy-updating algorithm for optimization problems. "Monotone" means that every iteration of the algorithm results in a global improvement of the value function. A brief exposition of the algorithm would be as follows. Consider again the optimization problem at the beginning of this section. Find

$$\inf_u \int_{t_0}^{T} L(x, u; t)\ dt + \phi(x_T)$$

on the trajectories of the system

$$\dot{x} = f(x, u; t) \qquad x(t_0) = x_0$$

Assume that the piecewise continuous control law $u_i(x, t)$ gives rise to the differentiable value function $V_i(x, t)$. If $x(t)$ is the resulting trajectory then $V_i(x(t), t)$ is a function of t and we have

$$\frac{d}{dt} V_i = - L(x, u_i; t) = \frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial x} f(x, u_i; t) \tag{3.40}$$

Consider now the control law

$$u_{i+1}(x, t) = \arg\min \left[L(x, u, t) + \frac{\partial V_i}{\partial x} \cdot f(x, u; t) \,\middle|\, u \in \Omega \right] \tag{3.41}$$

Assume that $u_{i+1}(x, t)$ is piecewise continuous and defines a value function $V_{i+1}(x; t)$, which also satisfies an equation similar to (3.40).

$$\frac{\partial V_{i+1}}{\partial t} + L(x, u_{i+1}; t) + \frac{\partial V_{i+1}}{\partial x} \cdot f(x, u_{i+1}; t) = 0 \tag{3.42}$$

Furthermore by (3.40) and the definition of u_{i+1}

$$\frac{\partial V_i}{\partial t} + L(x, u_{i+1}; t) + \frac{\partial V_i}{\partial x} f(x, u_{i+1}; t) \le 0 \tag{3.43}$$

Subtracting now (3.42) from (3.43) we get

$$\frac{\partial (V_i - V_{i+1})}{\partial t} + \frac{\partial (V_i - V_{i+1})}{\partial x} f(x, u_{i+1}; t) \le 0$$

and hence

$$\frac{\partial (V_i - V_{i+1})}{\partial t} + \min_u \frac{\partial (V_i - V_{i+1})}{\partial x} f(x, u; t) \le 0 \tag{3.44}$$

Now (3.44) is a HJB inequality corresponding to an optimization problem with payoff $L = 0$. Hence by Proposition 3.13 $V_i \ge V_{i+1}$ which shows that the control law u_{i+1} is better than u_i provided that it is admissible, i.e., piecewise continuous. If all assumptions hold, the above computational procedure yields a sequence of value functions $V_0 \ge V_1 \ge V_2 \ge \ldots$ which are monotone decreasing.

The above policy-iteration algorithm would provide an effective way to solve any optimization problem provided that the sequence $V_i(x, t)$ converges to the optimal value function $V(x, t)$. This is not true in general. However, it is true for the linear quadratic problem, which we examine in detail in the next section. First we would like to examine the application of Bellman's algorithm to the stability constrained infinite duration problems.

Find

$$V(x_0) = \inf_u \int_{t_0}^{\infty} L(x,u)\, dt \qquad \lim_{T\to\infty} x_T = 0$$

$$\dot{x} = f(x,u); \qquad x(t_0) = x_0 \tag{3.45}$$

Let us assume that the stabilizing control law $u_i(x,t)$ results in the value function $V_i(x)$. As in the proof of (3.40), it can be easily shown that

$$L(x,u_i) + \frac{\partial V_i}{\partial x} f(x,u_i) = 0 \tag{3.46}$$

Let us assume furthermore that the control law obtained from Bellman's algorithm is admissible, i.e. piecewise continuous and stabilizing

$$u_{i+1}(x,t) = \arg\min_u \left[L(x,u) + \frac{\partial V_i}{\partial x} f(x,u) \right] \tag{3.47}$$

and it gives rise to a value function $V_{i+1}(x)$. We will show that $V_{i+1}(x) \le V_i(x)$. In fact

$$V_{i+1}(x) = \int_{t_0}^{\infty} L(x,u_{i+1})\, dt = \int_{t_0}^{\infty} \left\{ L(x,u_{i+1}) - \frac{d}{dt} V_i(x) + \frac{\partial V_i}{\partial x} f(x,u_{i+1}) \right\} dt$$

$$= V_i(x) - V_i(x_\infty) + \int_{t_0}^{\infty} \left[L(x,u_{i+1}) + \frac{\partial V_i}{\partial x} f(x,u_{i+1}) \right] dt \tag{3.48}$$

We make the further assumption that $V_i(0) = V_{i+1}(0) = 0$. The integrand in (3.48) is nonpositive by the definition of u_{i+1} and (3.46). Hence

$$V_{i+1}(x) \le V_i(x) .$$

The above shows that Bellman's algorithm will be monotonically convergent even in the infinite duration, stability constrained case provided that we can guarantee that the control laws generated at each step of the algorithm are admissible, i.e. stabilizing. This happens to be the case in the most important such algorithm, Bellman's algorithm for the linear quadratic minimization problem which we examine next.

4.2 Algorithms for the Linear Regulator Problem

The application of the policy iteration algorithms to the regulator problem. Find

$$\inf \int_0^\infty x'Qx + u'u \quad dt; \quad \lim_{T\to\infty} \; x_T = 0 \tag{3.49}$$

$$\dot{x} = Ax + Bu \qquad x(0) = x_0\,; \quad (A, B) \text{ controllable}$$

is particularly enlightening. It hinges on the fact that the optimal value function is quadratic in the state, and hence the algorithm proceeds by updating the parameters of a quadratic form in the way stipulated by Bellman's algorithm. Starting from a stabilizing control law $u_0 = L_0 x$ i.e. such that $\text{Re}\,\lambda(A + BL_0) < 0$, the resulting value is $V_0(x) = x_0' K_0 x_0$ where

$$(A + BL_0)'K_0 + K_0(A + BL_0) = -Q - L_0'L_0$$

The policy minimization step suggests to use

$$u_1 = \arg\min_u \{xQx + u'u + 2x'K_0(Ax + Bu)\}$$

$$\Rightarrow u_1 = -B'K_0 x$$

In general, the updating procedure is

$$u_{i+1} = -B'K_i x \tag{3.50}$$

$$(A - BB'K_i)'K_{i+1} + K_{i+1}(A - BB'K_i) = -Q - K_i BB'K_i \tag{3.51}$$

The algorithm in equations (3.50) and (3.51) was derived by Puri and Gruver [6] from policy iteration considerations and by Kleinman [5] who derived it by applying Newton's root finding procedure to the Riccati equation

$$A'K + KA + Q = KBB'K \tag{3.52}$$

In both approaches, the crucial step of the algorithm is to show that the updating of the control laws is valid, i.e. it gives a stable control law. The arguments presented to that effect in both references are vague. We thus propose to give independent proofs, and in the process provide a characterization of the class of problems for which this algorithm can be successfully applied.

We first consider the case where the cost matrix Q is positive definite.

PROPOSITION 3.53: Consider the regulator problem (3.49) with $Q = Q' > 0$. Then

i) There exists a unique solution $K^+ > 0$ to

$$A'K + KA + Q = KBB'K \tag{3.54}$$

such that $\operatorname{Re} \lambda(A - BB'K^+) < 0$. The control law $u = -B'K^+x$ is the solution (3.49) and $V(x) = x'K^+x$ is the optimal value function.

ii) The sequence $K_0, K_1, \ldots, K_n, \ldots$ defined by equations (3.50), (3.51) is well defined, monotonically decreasing and converges to K^+. Each $u_i = -B'K_ix$ is a stabilizing control law.

<u>Proof.</u> Part (i) is well known [4] so for part (ii) we proceed by induction. Let $u = -Lx$ be a stabilizing control law and let $V(x) = x'Kx$, $K = K'$ be the corresponding value function. The assumption $Q > 0$ guarantees that $K > 0$. Also, K satisfies the Lyapunov equation

$$(A + BL)'K + K(A + BL) = -Q - L'L \tag{3.55}$$

Consider now the updated control law

$$u_1 = -B'Kx$$

and the system

$$\dot{x} = (A - BB'K)x \tag{3.56}$$

We now try $V = x'Kx$ as a Lyapunov function for (3.56). We have along the trajectories

$$\frac{d}{dt} V = x'[(A - BB'K)'K + K(A - BB'K)]x$$

$$= x'[-2K\,BB'K - Q - L'L + KBL + L'B'K]x$$

$$= x[-KBB'K - Q - (L - B'K)'(L - B'K)]x < 0$$

The derivative is nonpositive, and in fact it is negative for $x \neq 0$. Thus the Lyapunov function $V(x) = x'Kx$ implies the stability of (3.56).
The iteration implied in (3.50), (3.51) is admissible and gives rise to a monotone decreasing sequence of matrices K_i, and $K_i \geq K^+$. Hence

$\lim_{i \to \infty} K_i = \widetilde{K}$ exists and satisfies (3.54), as can be seen by taking limits in (3.51). Thus $\widetilde{K} \geq K^+ > 0$, which contradicts the uniqueness of the positive definite solution of (3.54) unless $\widetilde{K} = K^+$. Hence $\lim_{i \to \infty} K_i = K^+$.

Q. E. D.

The assumption $Q > 0$ is fairly restrictive. Linear quadratic problems in an indefinite Q will arise in the problems of stabilization and game theory to be considered in Sections IV and V. The algebraic Riccati equation

$$A'K + KA + Q = KBB'K \tag{3.57}$$

will play a crucial role in their solution. It becomes therefore important to establish the validity of the Bellman algorithm for the solution of the (ARE) (3.57) with a general Q matrix. Computational experiments, performed on a Wang-2200 minicomputer, have shown that the algorithm almost always converges monotonically to a solution of the (ARE) (3.57) (provided of course that a solution to (3.57) exists.) We can give a theoretical justification of these experimental results. The numerical experiments are given in Appendix 2. To prove the monotonic convergence of the algorithm we make strong use of the theory of the solutions of the algebraic Riccati equation. A summary of that theory is provided in Appendix 1. In fact our main result requires a mild condition on the maximal and the minimal solutions, K^+ and K^- of the Riccati equation (Theorem A.1.1 of Appendix 1). This result is stated in

PROPOSITION 3.58: Assume that (A, B) is controllable, and that there exists a real symmetric solution of the (ARE)

$$A'K + KA + Q = KBB'K \tag{3.57}$$

Assume furthermore that $K^+ > K^-$. Then Bellman's algorithm as expressed in equations (3.50), (3.51) gives rise to a decreasing sequence $K_1 > K_2 > \ldots > K_i > \ldots$ which converges to K^+ provided of course that the algorithm is started with a stabilizing control $u_0 = Lx$, i.e. $\mathrm{Re}\,\lambda(A + BL) < 0$.

To prove Proposition 3.58 we need the following lemmas

LEMMA 3.59: Let K be a real symmetric solution to the (ARE) (3.57) and let

$$V = \int_0^T x'Qx + u'u \quad dt$$

with $\dot{x} = Ax + Bu$, $x(0) = x_0$, $x(T) = x_T$. Then

$$V = \int_0^T \|u + B'Kx\|^2 \quad dt + x_0'Kx_0 - x_T'Kx_T \tag{3.60}$$

Proof. This is a special case of the well known lemma of "completing the square", for instance Lemma 6 of [9].

The following lemma states that Bellman's procedure for improving a given control law is effective even when the improved control law is applied for a subinterval of the entire optimization horizon. In particular, consider the infinite horizon cost functional of problem (3.49)

$$J(u;x_0) = \int_0^{\infty} x'Qx + u'u \quad dt$$

$$\dot{x} = Ax + Bu \qquad x(0) = x_0$$

and assume that the stabilizing control law $u_0 = Lx$ is applied. Then

$$J(Lx;x_0) = x_0'K_0x; \qquad K_0 = K_0'$$

where K_0 satisfies

$$(A + BL)'K_0 + K_0(A + BL) = -Q - L'L$$

Consider now the improved control law

$$u_1(x) = \arg\min_u \{x'Qx + u'u + 2x'K_0(Ax + Bu)\} = -B'K_0x$$

and let us apply the control $u_1(x)$ on the time interval $[0, T]$ and the control $u_0(x)$ on (T, ∞), where T is a fixed but arbitrary time. Call

this piecewise linear law $u_T(x)$ and $J(u_T; x_0)$ the value that results when it is applied. We claim that u_T is an improvement on u_0 regardless of T.

LEMMA 3.61: $J(x_0, u_T) \le J(x_0, u_0) = x_0' K_0 x_o$.

<u>Proof.</u> To simplify the notation let $V(x) = J(x; u_0)$. Along the trajectories of

$$\dot{x} = Ax + Bu_0$$

$V(x_t)$ becomes a time function and

$$\frac{d}{dt} V(x) = - xQx - u_0' u_0 = \frac{\partial V}{\partial x} (Ax + Bu_0)$$

and hence

$$x'Qx + u_0' u_0 + \frac{\partial V}{\partial x} (Ax + Bu_0) = 0 \tag{3.62}$$

Since $u_1(x)$ is defined as the minimum of the above expression, it follows that

$$x'Qx + u_1' u_1 + \frac{\partial V}{\partial x} (Ax + Bu_1) \le 0 \tag{3.63}$$

Consider now $J(u_T; x_0)$. We have

$$J(x_0; u_T) = \int_0^T x'Qx + u_1' u_1 \, dt + \int_T^\infty x'Qx + u_0' u_0 \, dt$$

$$+ \int_0^T \left[- \frac{d}{dt} V(x) + \frac{\partial V}{\partial x} (Ax + Bu_1) \right] dt$$

The second integral equals $V(x_T)$ *and the term*

$$\int_0^T -\frac{d}{dt} V(x)\ dt = V(x_0) - V(x_T).$$

Hence

$$J(x_0; u_T) = V(x_0) + \int_0^T \left[x'Qx + u_1'u_1 + \frac{\partial V}{\partial x}(Ax + Bu_1)\right] dt \quad .$$

From (3.63) we conclude that the integrand is nonpositive and hence

$$J(x_0; u_T) \leq V(x_0) = J(x_0; u_0) \qquad \text{Q.E.D.}$$

<u>Proof of Proposition 3.58</u>. It suffices to show that

$$u_{i+1} = \arg\min_u \{x'Qx + u'u + 2x'K_i(Ax + Bu)\}$$

gives rise to a stable control law. It would then follow that the sequence K_i defined in (3.51) is nonincreasing and bounded below by K^+. Hence $\lim_{i\to\infty} K_i = \widetilde{K} \geq K^+$ and $\widetilde{K}$ is a solution of (3.57), as can be seen by taking limits in (3.51). This contradicts the maximality of K^+ and hence $\lim_{i\to\infty} K_i = \widetilde{K} = K^+$.

To show that the control improving procedure results in stable control laws we consider as in the remarks preceding Lemma 3.61, the stabilizing control $u_0 = Lx$, the updated control $u_1 = -B'Kx$ and the piecewise linear law $u_T(x)$. We showed in Lemma 3.61 that $J(x_0; u_T) \leq x_0'Kx_0$. We now bound $J(u_T; x_0)$ from below as follows. Let

$$J(x_0; u_T) = \int_0^T xQx + u_1'u_1\ dt + \int_T^\infty x'Qx + u_0'u_0\ dt \qquad (3.64)$$

From Theorem A.1.4

$$\int_T^\infty x'Qx + u_0'u_0 \quad dt \geq x_T K^+ x_T \tag{3.65}$$

From Lemma 3.59

$$\int_0^T x'Qx + u_1'u_1 \quad dt = x_0'K^- x_0 - x_T'K^- x_T + \int_0^T \|u_1 + B'K^- x\|^2 \quad dt \tag{3.66}$$

In view of (3.65) and (3.66) we obtain from (3.64)

$$J(x_0; u_T) \geq x_0'K^- x_0 + x_T'(K^+ - K^-)x_T + \int_0^T \|u_1 + B'K^- x\|^2 \quad dt$$

and finally using the bound $J(x_0; u_T) \leq x_0'Kx_0$

$$x_0'(K-K^-)x_0 \geq x_T'(K^+ - K^-)x_T + \int_0^T \|u_1 + B'K^- x\|^2 \quad dt$$

and

$$x_0'(K - K^-)x_0 \geq x_T'(K^+ - K^-)x_T \tag{3.67}$$

Inequality (3.67) holds for any T. The assumption $K^+ - K^- > 0$ implies that $\|x_T\|$ is bounded for all T and thus the control law $u_1(x) = - B'Kx$ can't give rise to unbounded trajectories.

To show that u_1 is a stable law it remains to show that $A - BB'K$ can't have imaginary eigenvalues.* If it did, the linear system $\dot{x} = (A - BB'K)x$ would have a periodic solution for some initial condition $x(0) = \tilde{x}_0$. Let the period of that solution be T_1. Consider now our

*The case of a zero eigenvalue can be treated similarly.

minimization problem with initial condition $\tilde{x}_0$ and let the piecewise linear control $u_T(x)$ be applied with T an integral multiple of the period T_1, say $T = nT_1$. In the proof of Lemma 3.61 we proved the expression

$$J(\tilde{x}_0; u_T) = V(\tilde{x}_0) + \int_0^T \left[x'Qx + u_1'u_1 + \frac{\partial V}{\partial x}(Ax + Bu_1) \right] dt$$

where the integrand is a nonpositive quantity.

From the periodicity assumption this expression can be written as

$$J(\tilde{x}_0; u_T) = V(\tilde{x}_0) + n \int_0^{T_1} \left[x'Qx + u_1 u_1 + \frac{\partial V}{\partial x}(Ax + Bu_1) \right] dt$$

We claim that the integral has to be zero. Otherwise it would be a negative number and since n is arbitrary, $J(x_0; u_T)$ could be made smaller than $\tilde{x}_0' K^+ \tilde{x}_0$ for sufficiently large n, contradicting the fact that $\inf_u J(x; u) = x'K^+x$ (Theorem A.1.4). The integral is therefore zero and so is the integrand

$$x'Qx + u_1'u_1 + \frac{\partial V}{\partial x}(Ax + Bu_1) = 0$$

(Here we use the fact that the integrand is nonpositive). We know that u_1 is the unique control that minimizes the expression

$$x'Qx + u'u + \frac{\partial V}{\partial x}(Ax + Bu)$$

Also, the above expression becomes zero for $u = u_0$ as was shown in the proof of Lemma 3.61. The minimum is thus attained by u_0 and hence, by the uniqueness of the minimum, $u_1(x) = u_0(x)$ for all x on the periodic trajectory of $\dot{x} = Ax + Bu_1(x)$. The system $\dot{x} = Ax + Bu_0(x)$ gives rise

to the same periodic trajectory as $\dot{x} = Ax + Bu_1(x)$ if started at the initial condition $x(0) = \tilde{x}_0$. This is a contradiction to the assumption that u_0 is a stabilizing control. Q. E. D.

Remark. Appendix 1 shows that if $K^+(Q_0)$, $K^-(Q_0)$ are the stabilizing and destabilizing solution of the (ARE) (3.57)

$$A'K + KA + Q_0 = KBB'K$$

and $K^+(Q_0) \not> K^-(Q_0)$, a slight perturbation of Q_0 suffices to make the strict inequality, between K^+, K^- hold. Namely let $Q = Q_0 + \epsilon I$. It is easy to show that $K^+(Q_1)$, $K^-(Q_1)$ exist and

$$K^+(Q_1) > K^-(Q_1)$$

for any $t > 0$. It is in this sense that we can make the statement that Bellman's algorithm converges for almost any Riccati equation that has a solution.

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IV. STABILITY OF INTERCONNECTED SYSTEMS

1. Introduction

When confronted with the task of analyzing a complicated system, one can usually achieve a better understanding by considering it as an interconnection of simple subsystems. This is perhaps a natural way of understanding a man-made system since the engineering design process is mostly one of interconnecting components to achieve a system with desired behavior [1]. On the other hand, it is usually not obvious how the behavior of a collection of subsystems gives rise to a qualitatively different behavior of the overall system. The fact that we have little intuition about the behavior of an arbitrary interconnection of two systems might lead us to believe that it is impossible in general to study large systems by first analyzing their subsystems and then somehow putting the pieces together into information about the overall system. A major task of large scale system theory is therefore to respond to the above objections by identifying classes of systems for which interesting questions can be answered by studying their subsystems and their interconnections on a separate basis.

Part of Section II dealt with such an approach to the stability of interconnected systems, the main step of which was in postulating interconnection constraints that make possible the formation of a system Lyapunov function from the subsystem ones. The postulated interconnection constraints were to a great extent ad hoc, since it is hard to find any reason for postulating them except in order to make the desired results go through. In this Section we examine a more natural approach to the study

of stability of interconnected systems due to J. C. Willems [2], [3], [4], the dissipative system approach. We have called this approach natural because the fundamental definition, that of a dissipative system, formalizes the notion of a lossy physical system such as a passive electric network. It is thus expected that a large class of systems will satisfy the conditions for dissipativeness. The theory applies nicely to large systems: A system will be dissipative if it is the interconnection of dissipative subsystems and thus the property of dissipativeness is transmitted from lower hierarchical levels to higher ones.

In 2 we consider the dissipativeness of linear systems and that of their interconnection. It is conceivable though that not all subsystems forming an interconnected system will be dissipative. We thus address in 3 the problem of transforming such subsystems to dissipative ones through linear feedback on the subsystem state only. This process is referred to as decentralized stabilization and is reasonable when it is very costly to observe the state of a subsystem for the purpose of controlling a different, remotely located subsystem. The design of such decentralized controllers would involve an awkward search procedure. As an alternative we propose in 3.2 a class of linear controllers motivated from linear quadratic game theory. These controllers can be obtained by a straightforward numerical procedure based on the Bellman algorithm (hence referred to as B-algorithm) for solving a Riccati equation that was presented in Section III, Prop. 3.59. The justification of these techniques is given in 4.3. An argument to the effect that the proposed class of linear controllers is sufficient for stabilization purposes is given in 4.4. Finally the introduction of subsystem state observers is discussed in 4.5.

2. Dissipative Interconnected Systems

2.1 Preliminaries

A dynamical system in state-space form is defined [3] by

$\Sigma = \{U, \mathscr{U}, Y, \mathscr{Y}, X, \phi, r\}$ where

U = input alphabet

$\mathscr{U}$ = input space; a class of functions mapping R^+ into U

Y = output alphabet

$\mathscr{Y}$ = output space, analogous to $\mathscr{U}$

X = state space

ϕ = *state transition function*

r = readout function

The state transition function $\phi : R_+^2 \times X \times \mathscr{U} \to X^{(*)}$ satisfies the usual state transition axioms

(i) $\phi(t_0, t_0, x_0, u) = x_0$

(ii) If $u_1(t) = u_2(t)$ for $t_0 \le t \le t_1$ then $\phi(t_1, t_0, x, u_1) = \phi(t_1, t_0, x, u_2)$

(iii) $\phi(t_2, t_1, \phi(t_1, t_0, x, u), u) = \phi(t_2, t_0, x, u)$

The read-out function $r : X \times U \times R_+^2 \to Y$ determines the input-output structure of the system. It is required that $r(\phi(t, t_0, x, u), u(t), t, t_0)$ defined for $t \ge t_0$, be the restriction to $[t_0, \infty)$ of an element of $\mathscr{Y}$ denoted by $y(x, t_0, u)$.

In order to introduce the notion of a dissipative system let us assume that in addition to a dynamical system Σ we are given two scalar

---

$^*R_+^2 = \{(t_1, t_0) | t_1 \ge t_0\}$

functions $S : X \times R \to R$ and $w : U \times Y \times R \to R$. The function $S(x,t)$ has the interpretation of "energy" content of state x at time t and is called storage function. The function $w(u,v,t)$ has the interpretation of power supplied to the system by the external world and is thus called the supply rate. The supply rate depends on the instantaneous values of the system's input and output but not on the state. This definition reflects the notion that energy can only be transferred to a system through its external terminals and the transfer is not affected by the inner workings of the system, i.e. its state. Consider now a transition of Σ from (x_0, t_0) to (x_1, t_1) under the influence of an input $u(t)$. An amount

$$\int_{t_0}^{t_1} w(u(t), y(t), t) \quad dt$$

of energy is externally supplied to the system during its transition. If the inequality

$$S(x_0, t_0) + \int_{t_0}^{t_1} w(u(t), y(t), t) \ dt \geq S(x_1, t_1) \tag{4.1}$$

holds, it means that energy was lost in the transition since the storage at the new state is less than that at the old state plus the externally supplied energy. A system $\{\Sigma, w, S\}$ is called dissipative if (4.1) is satisfied for for every possible transition and in addition the storage function is nonnegative. The Dissipation Inequality (DI for short) (4.1) is invariant under addition of a constant to S and hence the nonnegativity of S can be satisfied if and only if S has a finite lower bound.

The relation of dissipative systems to stability hinges on the fact that if there is no external supply to the system, it will dissipate its

stored energy. In terms of the previous definitions, if an input $u(t)$ is applied such that $w(u(t), y(t), t) \le 0$ the (DI) implies that $S(x(t), t)$ will be decreasing in t and can serve as a Lyapunov function from which stability results will follow if S satisfies the requirements of any Lyapunov stability theorem.

One may wonder what is the advantage of checking the rather involved conditions for dissipativeness if one is only interested about the stability of a system for a certain class of inputs. It should be pointed out first that checking the dissipativeness condition does not involve choosing both a supply rate w and a storage function S. For a given supply rate w, it can be shown that if the nonnegative expression

$$S_a(x_0, t_0) \triangleq \sup_{\substack{t_1 \ge t_0 \\ x_0 \to}} - \int_{t_0}^{t_1} w(t)\, dt \tag{4.2}$$

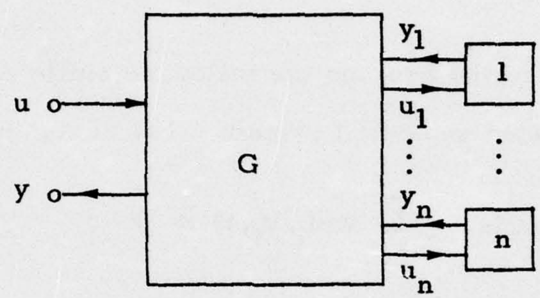
is defined, i.e. $S_a(x_0, t_0) < \infty \ \forall\, x_0, t_0$, S_a can serve as a storage function. The sup in (2) is taken over all transitions of Σ in (t_0, t_1) starting at x_0 at t_0, without any restriction on t_1 or the input used in (t_0, t_1). In fact it is shown in [3] that S_a is the smallest(*) possible storage function. This result shifts the burden of stability proofs to the choice of an external supply function w, a task about which one might have greater intuition than in choosing a Lyapunov function.

Perhaps the most important feature of the theory is the ease it can be applied to an interconnection of dissipative systems [2]. Let us assume that we are given a family of dynamical systems $\Sigma_i = \{U_i, \mathcal{U}_i, Y_i, \mathcal{Y}_i, X_i, \phi_i, r_i\}$ $i \in I$. The systems are then interconnected in some way to form a new system with inputs $\{U, \mathcal{U}\}$ and outputs $\{Y, \mathcal{Y}\}$. A memoryless

*If $S \ge 0$ satisfies (4.1), then $S \ge S_a$.

interconnection can be visualized as

Figure 1



i.e. as a function $G : U \times Y_1 \times \ldots Y_n \to Y \times U_1 \times \ldots \times U_n$. For the system in Fig. 1 to make sense one must somehow verify that an input $u \in \mathscr{U}$ gives rise to unique "outputs" $u_i \in \mathscr{U}_i$, $y_i \in \mathscr{Y}_i$, $y \in \mathscr{Y}$. This property may be quite hard to verify for a general interconnection, but it obviously holds for the type of subsystems to be considered here. So let us assume that the system in Fig. 1 is well defined and denote it by $X_{i\in I}\ \Sigma_i | G$. Assume furthermore that every Σ_i is dissipative with respect to a supply rate w_i and a storage function S_i, $i \in I$. The dissipativeness property is transferred to the interconnected system: Namely the system $X_{i\in I}\ \Sigma_i | G$ is dissipative with respect to the supply rate

$$w(u, y, t) \overset{\text{def}}{=} \sum_{i\in I} w_i(u_i, y_i, t) \tag{4.3}$$

and the storage function

$$S(x_{i\in I}, t) \overset{\text{def}}{=} \sum_{i\in I} S_i(x_i, t) \tag{4.4}$$

The proof of this fact is immediate [3, Prop. III-1]. It should be noted that for the definition (4.3) to make sense it must be possible to express the u_i, y_i's in terms of u, a fact that is guaranteed by the assumed well posedness of the problem and the memoryless property of the interconnections.

The above remarks provide the following sufficient conditions for the stability of an interconnected system. Let us assume that

$$w(u, y, t) = \sum_{i \in I} w(u_i, y_i, t) \leq 0$$

Then $S = \Sigma S_i$ is a Lyapunov function for $X_{i \in I} \Sigma_i | G$ since it decreases along the trajectories and stability results will follow if $\Sigma_{i \in I} S_i$ has further properties for Lyapunov theorems to apply.

2.2 Stability Conditions for Systems with Quadratic Supply Rates

The inequality

$$w(u, y, t) \leq 0 \tag{4.5}$$

guarantees that the storage function $S(t)$ of a dissipative system decreases during the application of the input u. Furthermore, if the storage function is positive definite, that is $\phi_1(\|x\|) \leq S(x, t) \leq \phi_2(\|x\|)$ where ϕ_i are scalar positive definite functions, $\phi(\infty) = \infty$, inequality (4.5) guarantees that $\|x_t\|$ is bounded. Furthermore if $w(u, y, t) \leq - \phi_3(\|x\|)$, global asymptotic stability follows. In particular if a system is dissipative w.r.t.

$$w(u, y) = u'u - \rho y'y \qquad \rho > 0 \tag{4.6}$$

and $S_a(x, t)$ is positive definite, the bound $u'u \leq \rho y'y$ guarantees bounded

trajectories, and $u'u \leq \rho\ y'y - \phi_3(\|x\|)$ asymptotically stable ones.

The study of linear systems with quadratic supply rates has received a good amount of attention. Consider the system

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}\right\} \tag{4.7}$$

The system will be dissipative with respect to the supply rate w of (4.6) if and only if

$$S_a(x_0, t_0) = \sup_{\substack{x(t_0)=x_0 \\ t_1 \geq t_0}} - \int_{t_0}^{t_1} (u'u - \rho\, x'C'Cx)\ dt$$

exists for all x_0, t_0. *Due to the time invariance of (4.7) it suffices to* consider $t_0 = 0$ and we write $S_a(x_0)$ instead of $S_a(x_0,t_0)$. Furthermore, it can be easily shown that

$$\sup_{x(0)=x_0} - \int_0^{t_1} w(t)\ dt \leq \sup_{x(0)=x_0} - \int_0^{t_2} w(t)\ dt$$

for $t_1 \leq t_2$ by noticing that a null input $u(t) = 0$ for $t \in [t_1, t_2]$ concatenated to any input $u(t)$ for $t \in [0, t_1)$ will result in

$$- \int_0^{t_1} w(t)\ dt \leq - \int_0^{t_2} w(t).$$

Hence

$$- S_a(x_0) = \inf_{x(0)=x_0} \int_0^{\infty} \{- \rho\ x'C'Cx + u'u\}\ dt \tag{4.8}$$

Theorem A. 2 of Appendix 1 shows that

$$S_a(x_0) = -x_0'K^+x_0 \tag{4.9}$$

exists (and hence (4. 7) is dissipative) if and only if the following (FDI) holds for $\omega \in R$

$$I - \rho B'(-j\omega I-A')^{-1}C'C(j\omega I-A)^{-1}B \geq 0 \tag{4.10}$$

Now $S_a(x)$ is a positive definite function and the additional condition $w = u'u - \rho x'C'Cx \leq 0$ guarantees the boundedness of the trajectories of (4. 7). Global asymptotical stability follows if $u'u \leq \rho x'C'Cx - \phi(x)$.(*)

A further application of the above is to the stability of interconnected systems. Consider an autonomous system of the form

$$\dot{x}_i = A_ix_i + B_ih_i(x_1, \ldots, x_N; t) \qquad i = 1, \ldots, N \tag{4.11}$$

where $x_i \in R^{n_i}$, B_i is a $n_i \times m_i$ full rank matrix and $h_i(x_1, \ldots, x_n; t) : R^{n_1+\ldots+n_N} \times R \to R^{m_i}$. The pairs (A_i, B_i) are controllable $\forall$ i. We can view this system as an interconnection of N input-output subsystems

$$\left.\begin{aligned} \dot{x}_i &= A_ix_i + Bu_i \\ y_i &= x_i \end{aligned}\right\} \tag{4.12}$$

with the interconnection constraint

$$u_i = h_i(y_1, \ldots, y_N; t)$$

If each input-output system in (4. 12) is dissipative with respect to a supply function

$$w_i(u_i, y_i) = u_i'u_i - \rho_i x_i'x_i \tag{4.13}$$

the interconnected system in (11) is also dissipative in the supply rate

*The weaker condition $u'u \leq (\rho - \epsilon)x'C'Cx$ guarantees gl. as. stability. See [6, Sect. 33].

$$w(t) = \sum_{i=1}^{N} \{u_i^{\prime} u_i - \rho_i x_i^{\prime} x_i\} \tag{4.14}$$

The supply rate (4.14) is denoted as $w(t)$ instead of $w(u, y, t)$ since the system is autonomous. In view of the remarks at the beginning of this section, boundedness of the trajectories interconnected system (4.11) is guaranteed if

$$w(t) = \sum_{i=1}^{N} \{u_i^{\prime} u_i - \rho_i x_i^{\prime} x_i\} \leq 0$$

or equivalently

$$\sum_{i=1}^{N} h_i^{\prime}(x_1, \ldots, x_N; t) h_i(x_1, \ldots, x_N; t) \leq \sum_{i=1}^{N} \rho_i x_i^{\prime} x_i \tag{4.15}$$

Inequality (4.15) is a condition on the interconnection of the system (4.11). It is a distinct improvement on the conditions that could be obtained without the application of the facts about the interconnection of dissipative systems. Each subsystem of (4.12) would be stable if $u_i^{\prime} u_i \leq \rho_i x_i^{\prime} x_i$ or

$$h_i^{\prime}(x_1, \ldots, x_N; t) h_i(x_1, \ldots, x_N; t) \leq \rho_i x_i^{\prime} x_i \tag{4.16}$$

If (4.16) is satisfied for all i the interconnected system (4.11) is stable. This is a much stronger stability condition than (4.15). In fact (4.16) is very restrictive since it implies that $h_i(x_1, \ldots, x_N)$ depends on x_j, $j \neq i$ in a very weak manner.

Similar remarks can be made for the interconnection of systems which are dissipative with respect to different supply rates. For the quadratic rate

$$w(u, y) = u'Py$$

the minimal nonnegative storage function can be obtained as

$$S_a(x_0) = \frac{1}{2} \sup_{\substack{t_1 \geq t_0 \\ x_0 \rightarrow}} - \int_{t_0}^{t_1} 2u'Py \; dt$$

which becomes, by making the same transformations as in the derivation of (4.8), and setting $y = x$

$$S_a(x_0) = -\frac{1}{2} \inf_{x_0 \rightarrow} \int_0^{\infty} 2u'Px \; dt$$

According to Theorem A.1, the infimum exists if and only if the (ARE)

$$A'K + KA = (KB + P')(B'K + P)$$

has a negative semi-definite solution. If it possesses a negative definite solution K^+,

$$S_a(x_0) = -\frac{1}{2} x_0' K^+ x_0$$

and if in addition K^+ is feedback stabilizing, the infimum is a minimum and is attained by the input $u = -(B'K^+ + P)x$.

Applying these results to the interconnected system (4.11) we obtain an analog to stability condition (4.15) as

$$\sum_{i=1}^{N} h_i(x_1, \ldots, x_N; t) P_i \, x_i \leq 0 \tag{4.17}$$

under the assumption that the subsystems in (4.12) are dissipative with respect to the supply rates $w_i(u_i, y_i) = u_i P_i x_i$. The $m_i \times n_i$ weighting

matrices P_i are unrestricted, as long as the subsystems are dissipative and should be chosen so as to facilitate the verification of (4.17).

In the remainder of this section we will be mainly concerned with supply rates of the form (4.13), i.e. $w(u, y) = u'u - \rho y'y$. Most of what will be said about them will apply to inner-product supply rates $w(u, y) = u'Py$ since the verification of dissipativeness in both cases involves algebraic Riccati equations. Furthermore an interconnection constraint involving norms seems more plausible than one involving inner products.

Given that the stability condition (4.15) becomes weaker with increasing ρ_i, we are interested in finding the largest possible values for ρ_i, $i = 1, \ldots, N$ which will then provide the best possible stability condition of the form (4.15). An equivalent problem is to determine the range of ρ such that the system

$$\dot{x} = Ax + Bu$$

is dissipative with respect to the supply rate $w(u, y) = u'u - \rho x'x$. Again this is equivalent to finding the range of ρ for which the (ARE)

$$A'K + KA - \rho I = KBB'K \tag{4.18}$$

has a negative definite solution, or finding those ρ for which the Frequency Domain Inequality (FDI)

$$I - \rho B'(-j\omega I - A')^{-1} (j\omega I - A)^{-1} B \geq 0 \tag{4.19}$$

holds for all ω. Numerical methods based on either the (ARE)(4.18) or the (FDI) (4.19) are possible in view of the fact that the range of ρ is a

closed interval $[0, \rho_0]$.* A good estimate for ρ_0 can be obtained by a half-interval search procedure. If ρ_0 is known to lie in an uncertainty interval $[a, b]$ a search at the half point $h = a+b/2$ will show whether ρ_0 is in the right or the left half of the interval according to whether the (ARE) (4.18) has or has not a negative definite solution. To ascertain the existence of a solution of (4.19) the B-algorithm presented in Section III can be used. We show later that it is applicable. We first present the entire algorithm for the determination of ρ_0 up to a desired accuracy ϵ.

Algorithm 1

Step 1. Let $a = 0$ and $b = M$, M an arbitrary positive number.

Step 2. Use the B-algorithms to determine whether (4.18) has a nondefinite solution for $\rho = b$. If it does, let $a \leftarrow b$, $b \leftarrow 2b$ and repeat this step. If it does not, proceed to Step 3.

Step 3. Upon exiting Step 2, we know that $\rho_0 \in [a, b]$. Let $h \leftarrow a+b/2$ and use the B-algorithm to see if (4.18) has a nondefinite solution for $\rho = h$. If it does set $a \leftarrow h$ and go to Step 4. If not, set $b \leftarrow h$ and go to Step 4.

Step 4. If $|a-b| < \epsilon$, the desired accuracy has been achieved and we stop. If not, repeat Step 3 with the new values of a, b.

---

*Proof. For a fixed ω, inequality (4.19) is satisfied for $\rho \in [0, \max \lambda\{B'(j\omega I - A')^{-1}(j\omega I - A)^{-1}B\}] = [0, \rho(\omega)]$, a closed interval. The range of ρ satisfying (4.19) for all ω is $\cap_{\omega \in R} [0, \rho(\omega)]$, which is a closed interval, being an intersection of closed sets.

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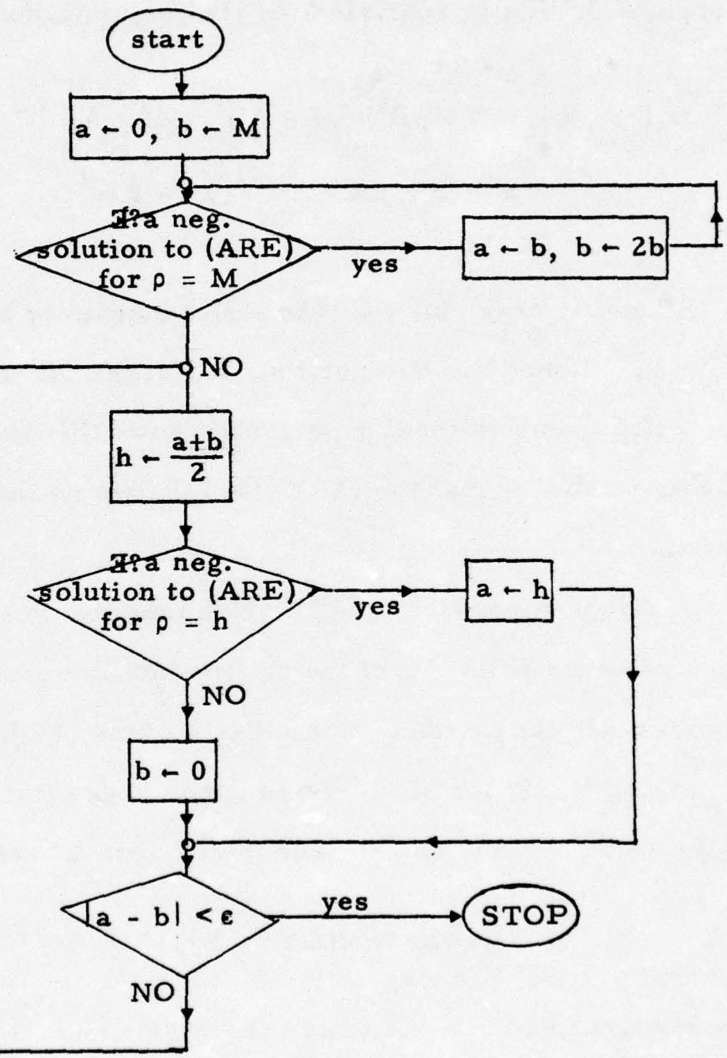
Figure 2

A Flowchart for Algorithm 1



The validity of Algorithm 1 depends on the convergence of the B-algorithm for $\rho < \rho_0$. Its convergence can be proven as follows. According to Proposition 3.58 we must show that $K^+ > K^-$. According to Theorem A.3, this is equivalent to showing that, for some $\epsilon > 0$,

$$\begin{aligned} H_\rho(-j\omega, j\omega) &= I - \rho B'(-j\omega I - A')^{-1}(j\omega I - A)^{-1}B \\ &\geq \epsilon\ B'(-j\omega I - A')^{-1}(j\omega I - A)^{-1}B \end{aligned}$$

Since $H_{\rho_0}(-j\omega, j\omega) \geq 0$ the strict inequality holds for $\rho < \rho_0$, $\epsilon = \rho_0 - \rho$. Hence the B-algorithm converges for $\rho < \rho_0$. If it is applied for $\rho > \rho_0$, computational experience shows that instability will soon occur thus showing that the (ARE) (4.18) does not have a nondefinite solution.

Although Algorithm 1 will work no matter what choice is made for the initial upper point M of the uncertainty interval (Step 1), a good choice for M can be made on the basis of the (FDI) (4.19). The maximum ρ satisfying (4.19) for $\omega = 0$ is an upper bound for ρ_0. The (FDI) becomes $I \geq \rho\ B'(AA')^{-1}B$ for $\omega = 0$ and the largest ρ satisfying it is

$$\rho = [\max\ \lambda\{B'(AA')^{-1}B\}]^{-1} = M_1 \quad .$$

We are guaranteed that the (ARE) (4.18) does not have a nondefinite solution for $\rho = M_1$ and thus ρ_0 is included in $[0, M_1]$. We can therefore start Algorithm 1 at Step 3 with $a = 0$, $b = M_1$.

Given that the (FDI) (4.19) is an algebraic condition, one would hope for an analytic method for computing ρ_0. Of course, a straightforward verification of the (FDI) (4.19) is inefficient even for a given ρ. For

special cases though it provides a superior way of finding ρ_0. Rewriting (4.19) as

$$I - \rho B'[\omega^2 I + A'A + j\omega(A-A')]^{-1} B \geq 0 \tag{4.20}$$

we note that for $A = A'$ *it simplifies to*

$$I - \rho B'[\omega^2 I + AA]^{-1} B \geq 0$$

The matrix $\omega^2 I + AA$ is positive definite and increasing in ω^2. For a fixed ρ, the minimum of the L.H.S. is at $\omega = 0$ and hence

$$\rho_0 = [\max \lambda\{B'(AA)^{-1}B\}]^{-1}$$

i.e. the bound M_1 is precisely, ρ_0.

We note finally that an inspection of (4.19) shows that it will definitely hold for ω large enough. We give here an estimate of how large is "large enough", an estimate that simplifies the task of a straightforward verification of (4.19). Consider the expression

$$\phi(\omega) = \omega^2 I + AA' + j\omega(A - A')$$

when $A \neq A'$. The matrix ϕ is Hermitian and its derivative with respect to ω is

$$\phi' = 2\omega I + j(A - A')$$

which is again Hermitian. It can be shown that ϕ' is positive definite for $\omega > \omega_0$ where

$$\omega_0 = \frac{1}{2}\{\max \lambda[(A - A')(A' - A)]\}^{1/2} \geq 0$$

and negative definite for $\omega < -\omega_0 \leq 0.$* Therefore ϕ is increasing for $|\omega| > \omega_0$ and equivalently, the L.H.S. of (4.20) decreases for $|\omega| > \omega_0$. In verifying that (4.20) or (4.19) holds for a fixed ρ one has to check (4.20) for $|\omega| < \omega_0$ rather than all $\omega \in R$. Even so, this is a much less efficient procedure than Algorithm 1.

3. Decentralized Stabilization

3.1 Linear Controllers

We consider now the interconnected control system

$$\dot{x}_i = A_i x_i + B_i u_i + h_i(x_1, \ldots, x_N; t) \quad i = 1, \ldots, N \tag{4.21}$$

with $x_i \in R^{n_i}$, $u_i \in R^{m_i}$ and $h_i : R^{n_1 + \ldots + n_N} \times R \to R^{n_i}$. The difference between this system and system (4.11) is that the latter is

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*Proof. Consider first a Hermitian matrix $M = P + jQ$, $P = P'$ and $Q = -Q'$. We get by direct computation that

$$(x' - jy')M(x + jy) = (x', y') \begin{pmatrix} P & Q' \\ Q & P \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The positive definiteness of M is equivalent to that of $\begin{pmatrix} P & Q' \\ Q & P \end{pmatrix}$.

Note that

$$\begin{pmatrix} I & 0 \\ -QP^{-1} & I \end{pmatrix} \begin{pmatrix} P & Q \\ Q' & P \end{pmatrix} \begin{pmatrix} I & -P^{-1}Q' \\ 0 & I \end{pmatrix} = \begin{pmatrix} P & 0 \\ 0 & P - QP^{-1}Q' \end{pmatrix}$$

Thus $M > 0 \Longleftrightarrow$ (i) $P > 0$ (ii) $P - QP^{-1}Q' > 0$. For $M = 2\omega I + J(A - A') = \phi'$ and $\omega > 0$ we get that $\phi' > 0 \Longleftrightarrow 2\omega I - ((A-A')(A'-A)/2\omega) > 0$ or $4\omega^2 I > (A-A')(A'-A)$. This can hold if and only if $\omega > \omega_0 = 1/2\{\max \lambda[(A-A')(A'-A)]\}^{1/2} \geq 0$. It can be similarly shown that $\phi' < 0$ for $\omega < -\omega_0 \leq 0$.

autonomous, while (4.21) admits inputs u_i, $i = 1, \ldots, N$. It is then possible to apply feedback to (4.21) in order to modify it in some desirable way. We assume here that a feedback controller on u_i can be a function of x_i or of an observable output $y_i = \gamma_i(x_i)$ of the i-th subsystem. In other words, it is too costly to implement a controller for the i-th subsystem who can observe the state of any subsystem *different* from i.

An essential feature of any controller is that it must lead to a stable system. In this and the next section we investigate the question of decentralized stabilization of system (4.21), i.e. the design of controllers $u_i = u_i(\gamma_i(x_i))$ which stabilize (4.21) in the presence of the interconnection terms h_i. Time-invariant controllers linear in the subsystem state are considered. We will assume that either the entire state x_i, $i = 1, \ldots, N$ or subsystem outputs $y_i = C_i x_i$ are available.

In the previous section we showed that a bound on the interconnections of the form

$$\sum_{i=1}^{N} h_i'(x_1, \ldots, x_N; t) h_i(x_1, \ldots, x_N; t) \le \sum_{i=1}^{N} \rho_i x_i' x_i \tag{4.15}$$

guarantees the stability of an interconnected system provided the subsystems are dissipative with respect to the supply rate $w_i(u_i, y_i) = u_i'u_i - \rho_i y_i' y_i$. This result might not be applicable to (4.21) if A_i is not stable for some $i \in 1, \ldots, N$ and $u_i = 0$. The i-th subsystem will not be dissipative for any ρ_i. On the other hand, if (A_i, B_i) is a controllable pair, a feedback $u_i' = P_i x_i$ can be chosen to make

$$A_i + B_i P_i \overset{\text{def}}{=} A_P$$

stable. The system

$$\dot{x}_i = A_P x_i + v_i \, ; \qquad y_i = x_i \tag{4.22}$$

will be dissipative with respect to $w_i(v_i, y_i) = v_i' v_i - \rho_i x_i' x_i$ provided the (FDI)

$$(\omega^2 - \rho)I + A_P' A_P + j\omega(A_P - A_P') \geq 0$$

is satisfied for all ω. Since the (FDI) holds for $\rho_i = 0$ it will hold for some sufficiently small $\rho_i > 0$. It follows therefore that by stabilizing each subsystem we can guarantee its dissipativeness for a small enough ρ_i and hence we can guarantee the stability of the overall subsystem for interconnections small enough to satisfy (4.15) for the above ρ_i's.

It is perhaps natural to conjecture that by placing the poles of a subsystem far into the left half plane it can be made dissipative with respect to any supply rate $w(u, y) = u'u - \rho\, y'y$ i.e., for any $\rho > 0$. This conjecture is false as can be seen in the following example.

Consider a system

$$\left.\begin{aligned} \dot{x} &= (A + BP)x + u \\ y &= x \end{aligned}\right\} \tag{4.23}$$

where $x, u \in R^n$, B is a $n \times m$ matrix, P a $m \times n$ one and (A, B) is controllable. P is to be chosen so that $\operatorname{Re} \lambda(A + BP) < 0$ and (4.23) is dissipative with respect to $w(u, y) = u'u - \rho x'x$ for a given ρ. We are interested in establishing the ρ for which the choice of such a P is possible. In particular consider

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (p, q) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right] + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad (4.24)$$

For this example,

$$A + BP = \begin{pmatrix} 0 & 1 \\ p & q \end{pmatrix}$$

and it is stable provided $p, q < 0$. *Furthermore*

$$(B, AB) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

so that (A, B) is controllable. For a given ρ the system (4.24) is dissipative provided

$$(\omega^2 - \rho)I + (A + BP)'(A + BP) + j\omega[(A + BP) - (A + BP)'] \geq 0$$

for all ω *or, after some* algebra, *if*

$$\begin{bmatrix} p^2 + (\omega^2 - \rho) & pq + j\omega(1 - p) \\ pq - j\omega(1 - p) & q^2 + 1 + (\omega^2 - \rho) \end{bmatrix} \geq 0 \qquad (4.25)$$

for all ω. The Sylvester criterion applied to (4.25) results to the inequalities

$$p^2 + (\omega^2 - \rho) > 0 \qquad (4.26)$$

$$[p^2 + (\omega^2 - \rho)]\,[q^2 + 1 + (\omega^2 - \rho)] - [p^2q^2 + \omega^2(1 - p)^2] > 0 \qquad (4.27)$$

for all $\omega \in R$. Inequalities (4.26) holds for all ω if and only if $p^2 > \rho$. Now, for a fixed ρ a straightforward computation shows that the minimum of the L.H.S. of (4.27) occurs at either $\omega_0^2 = \rho - p - q^2/2$ if $\rho - p - q^2/2 > 0$ or at $\omega_0^2 = 0$ if $\rho - p - q^2/2 \le 0$. In the first case, the L.H.S. of (4.27) for ω_0^2 becomes

$$\text{LHS} = -\rho - \left(p + \frac{q^2}{2}\right)^2 + p^2(1 - \rho) + 2p\rho \qquad (4.28)$$

and in the second case, $\omega = 0$

$$\text{LHS} = p^2(1 - \rho) + \rho^2 - \rho - \rho^2q^2 \qquad (4.29)$$

In either case it can be checked that for $1 - \rho < 0$ both expressions are negative.

This example shows that no matter what linear feedback is applied to (4.24), it can't be made dissipative for $\rho > 1$. The result is intuitively plausible. Consider again

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ p & q \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Note that for $u_1 = -x_2$ results in $\dot{x}_1 = 0$ and thus $x_1(0) = \text{const.}$ If $\rho > 1$ the supply rate $w(u, y)$ for $u_1 = -x_2$, $u_2 = 0$ is negative,

---

This is obvious for (4.28). As far as (4.29) is concerned, its supremum over $p^2 > \rho$ is at $p^2 = \rho$ and equals $-\rho q^2 < 0$.

in fact $w(u, v) = (\rho - 1)x_1^2(0) - \rho x_2^2(t) \leq (\rho - 1)x_1^2(0)$ for all t and thus

$$S_a(x_0) = \sup_{x_0 \to} - \int_0^\infty w(t) \, dt$$

is unbounded provided $x_1(0) \neq 0$. Thus (4. 24) is not dissipative for $\rho > 1$. The catch is that in (4. 24) linear feedback modifies only the $\dot{x}_2$ equation but not the $\dot{x}_1$ equation which can be made unstable even when $w(u, y) \leq 0$, i. e. $u'u \leq \rho\, x'x$ provided $\rho > 1$.

The above example shows that the determination of the possible ρ's for which a feedback controller $u = Px$ exists to make the system dissipative in the rate $w(u, y) = u'u - \rho\, y'y$ is a meaningful problem, i. e., we are not guaranteed that for any ρ such a P exists. In fact it can be shown that the set

$$\mathscr{R} = \left\{ \rho \;\middle|\; \exists \text{ a } P = P(\rho) \text{ such that the system } \dot{x} = (A + BP)x + u; \; y = x \text{ is dissipative in } w = u'u - \rho\, x'x \right\}$$

is bounded above if Rank $(B) < n$, n being the dimension of x. More precisely, R is an interval of the form $[0, \rho_0)$ or $[0, \rho_0]$ where $\rho_0 = \text{l. u. b. } \mathscr{R}$.

Determining ρ_0 is an important problem, as it provides a measure of our capacity to stabilize a nonlinear system with linear feedback control. In the above 2-dimensional example we showed by using the frequency domain approach that $\rho_0 \leq 1$. The determination of the exact value of ρ_0 by a frequency domain method would lead to a harder problem. In general, for a given ρ one would like to have an algorithm for determining a $n \times m$ matrix $P = P(\rho)$ such that

$$\left.\begin{array}{ll} \text{(i)} & \text{Re}\,\lambda(A + BP) \overset{\text{def}}{=} \text{Re}\,\lambda(A_P) < 0 \\ \text{(ii)} & I - \rho(-j\omega I - A_P')^{-1}(j\omega I - A_P)^{-1} \geq 0 \quad \forall\,\omega \end{array}\right\} \tag{4.30}$$

if such a P exists. In the next section we will provide such an algorithm, the motivation of which comes from the study of infinite duration games. The procedure will generate control matrices P by solving Riccati type equations by a numerical procedure similar to Algorithm 1. The convergence of the procedure will also be examined.

3.2 Game Theoretic Stabilization

Let us consider the system

$$\dot{x} = Ax + Bu + v \tag{4.31}$$

We want to design a linear controller $u = Px$ such that

$$\dot{x} = (A + BP)x + v \tag{4.32}$$

is dissipative for the rate $v'v - \rho\, x'x$, or equivalently

$$S_a(x_0) = -\inf_{v} \int_0^\infty \{-\rho\, x'x + v'v\}\, dt < \infty$$

$$\dot{x} = (A + BP)x + v \qquad x(0) = x_0$$

If such a controller exists, then the expression

$$\inf_{u=Px} S_a(x_0) = -\sup_{u=Px} \inf_{v} \int_0^\infty \{-\rho\, x'x + v'v\}\, dt \tag{4.33}$$

is finite. The form of (4.33) motivates us to consider the following class of problems in the positive parameters η, θ:

$$\left.\begin{aligned} J &= \sup_u \inf_v \int_0^\infty (-x'x - \theta u'u + \eta v'v)\, dt \\ \dot{x} &= Ax + Bu + v\,; \qquad x(0) = x_0 \end{aligned}\right\} \tag{4.34}$$

The understanding is that the $\inf_v$, $\sup_u$ are taken over non-dynamic closed loop strategies. In (4.34) the constraint $u = Px$ for some P is relaxed and the term $-\theta u'u$ has been added to the integrand. These changes have been made in the expectation that for a significant range of η J will be finite and the $\sup_u$ will be a minimum at some linear control $u = Mx$. If this is so, the following relation between (4.34) and (4.33) is evident

$$\begin{aligned} \inf_{u=Px} S_a(x_0) &= -\rho \sup_{u=Px} \inf_v \int_0^\infty -x'x + \frac{1}{\rho} v'v \quad dt \\ &\leq -\rho \sup_{u=Px} \inf_v \int_0^\infty -x'x - \theta u'u + \frac{1}{\rho} v'v \quad dt \\ &\leq -\rho \sup_u \inf_v \int_0^\infty -x'x - \theta u'u + \frac{1}{\rho} v'v \quad dt < \infty \end{aligned}$$

The finiteness of J in (4.34) for some η, θ implies that

$$\inf_{u=Px} S_a(x_0) < \infty$$

for $\rho = 1/\eta$ and in fact the system $\dot{x} = (A + BM)x + v$ will be dissipative in the rate $w = v'v - 1/\eta\, x'x$ and hence the condition $f'(x,t)f(x,t) \le 1/\eta\; x'x$ will guarantee stability. In the sequel we will provide an algebraic criterion for determining the finiteness of (4.34) for given parameters as well as a *procedure that will generate the* appropriate linear controller $u = Mx$.

The price one pays for the above simplification is a conservativeness in the estimate of the range of ρ, i.e. of $\mathscr{R}$. Indeed, problem (3.34) corresponds to determining the dissipativeness of a system in the supply rate $v'v - \rho(x'x + \rho\theta\, u'u)$ which is smaller than the original rate $v'v - \rho\, x'x$. For a fixed θ, the range of ρ (equivalently of η) for which (4.34) has a finite solution is strictly included in $\mathscr{R}$. This conservativeness can be removed by taking θ be small enough. We will show in the next section that if $\rho \in \mathscr{R}$ and $\rho < \rho_0 = \text{l.u.b.}\mathscr{R}$ then for all $\theta < \theta(\rho)$ and $\eta = 1/\rho$ the *inf-sup* in (4.34) will be finite. Thus if a stabilizing control exists, it can be found by solving (4.34) for θ sufficiently small. For ease of exposition we assume in the remainder of this section that $\theta = 1$ and return to the case $\theta \to 0$ in 3.4.

We would like to derive conditions for the existence of

$$
\begin{aligned}
J(x_0) &= \inf_u \sup_v \int_0^\infty x'C'Cx + u'u - \eta\, v'v \quad dt \\
\dot{x} &= Ax + Bu + Dv \qquad\qquad x(0) = x_0
\end{aligned}
\tag{4.35}
$$

where (A, C) is observable and (A, B), (A, D) controllable. This is a more general version of (4.34). The inf-sup problem (4.35) is intimately related to the game with value functional

$$\left.\begin{aligned} J(x_0; u, v) &= \int_0^\infty x'C'Cx + u'u - \eta\, v'v \quad dt \\ \dot{x} &= Ax + Bu + Dv\,; \qquad x(0) = x_0 \end{aligned}\right\} \tag{4.36}$$

where u is the minimizing player and v the maximizing player. This game is treated in detail in the next chapter. Here we present the results relevant to the inf-sup problem (4.35); proofs which are given in the next chapter are omitted.

PROPOSITION 4.1. A number p is the value of a game if $\inf_u \sup_v J = \sup_v \inf_u J = p$. Consider the game (4.36). Then

(a) A value of (4.36) in closed loop strategies exists if and only if the Algebraic Riccati Game equation (ARG)

$$A'K + KA + C'C = K(BB' - 1/\eta \;\; DD')K \tag{4.37}$$

has a real positive definite solution.

(b) If (4.37) has a positive definite solution for some η it has a <u>smallest</u> positive definite solution denoted by $K^+ = K^+(\eta)$. The value of (4.36) is

$$V(x_0) = x_0' K^+ x_0$$

Furthermore $\quad \mathrm{Re}\,\lambda(A - BB'K^+ + 1/\eta \;\; DD'K^+) \le 0.$

The inf-sup problem (4.35) is finite if and only if the game (4.36) has a finite value.

<u>Proof.</u> If a finite value exists the inf-sup is finite. However if a value does not exist for some initial condition x_0, it will be shown in the

next chapter that the maximizing player can guarantee an arbitrarily large value of the value functional, and thus $\inf_u \sup_v \geq \sup_v \inf_u \geq M$ for every M, i.e. the inf-sup does not exist. Q.E.D.

A further important property of the inf-sup (4.35) is that it is actually a min-sup, the min occurring at the controller $u = -B'K^+x$.

PROPOSITION 4.2. Assume that $K^+ = K^+(\eta)$ exists. Then the controller $u = -B'K^+x$ attains the inf-sup for problem (4.35).

<u>Proof.</u> The value of the inf-sup is, according to Proposition 4.1 $x_0'K^+x_0$. When $u = -B'K^+x$ we have

$$J = -\inf_v \int_0^\infty -x'(C'C + K^+BB'K^+)x + \eta \; v'v \quad dt$$

$$\dot{x} = (A - BB'K^+)x + Dv \qquad x(0) = x_0$$

The corresponding Riccati equation is

$$(A - BB'K^+)'M + M(A - BB'K^+) - (C'C + K^+BB'K^+) = \frac{1}{\eta} MDD'M$$

It can be verified by inspection that $M = -K^+$ is a negative definite solution satisfying

$$\text{Re}\,\lambda(A - BB'K^+ + 1/\eta \; DD'K^+) \leq 0$$

It follows from Theorem A.1.3 that the value of the infimum is $x_0'K^+x_0$.

Q.E.D.

3.3 Iterative Solutions of the Game Riccati Equation

We consider now the problem of establishing the range of η for which (4.35) is finite. This is equivalent to finding the η's for which the (ARG) (4.37) has a positive definite solution. As a first step we will transform the (ARG) to a Riccati equation for which the Bellman procedure we used in Algorithm 1 is again applicable. Let us assume that $K = K' > 0$ is a solution of (4.37). Then $M = K^{-1}$ is a solution of

$$- AM - MA' + (BB' - 1/\eta \; DD') = MC'CM \tag{4.38}$$

Conversely if (4.38) has a solution $M>0$, then $K = M^{-1} > 0$ is a solution of (4.37). The (ARE) (4.38) is of the type covered in Theorem 4.3 since the pair $(-A', C')$ is controllable, (A, C) being observable. Applying now Theorem 4.3 the (ARE) (4.38) we see that since it was shown to have a solution $M > 0$, it will possess in addition a pair of real symmetric solutions M^+, M^- such that

(a) $M^- \leq M \leq M^+ \quad \forall$ real symmetric solution

(b) $\operatorname{Re} \lambda(-A' - C'CM^+) \leq 0$

$\operatorname{Re} \lambda(-A' - C'CM^-) \geq 0$

Thus if $M > 0$ is a positive definite solution, M^+ will also be positive definite. Under the conditions of Proposition 3.58, i.e., $M^+ - M^- > 0$, the B-algorithm will converge to M^+ and a check of the positivity of the numerically computed M^+ suffices to determine whether $(M^+)^{-1}$ is a positive definite solution to the (ARG) (4.37).

The condition $M^+ - M^- > 0$ is not restrictive in the case where D is an $n \times n$ identity matrix, which is exactly the case of interest, i.e.,

problem (4.34). The (ARE) (4.38) becomes

$$- AM - MA' + (BB' - 1/\eta\ I) = MC'CM \tag{4.39}$$

It is clear that if (4.38) has a real symmetric solution for some η_1 it will have solutions for all $\eta \geq \eta_1$. In fact by remark similar to those used following the (FDI) (4.19) we conclude that (4.39) has solutions for $\eta \in [\eta_0, \infty)$ and for any $\eta \in (\eta_0, \infty)$ we have that

$$\text{Re}\ \lambda(- A' - CC'M^+) < 0 \quad {}^*$$

By Theorem 5.3 of [5] this is equivalent to $M^+ - M^- > 0$. We thus showed that the numerical procedure will converge for every $\eta < \eta_0$.

The above remarks have a further consequence. Let us assume that for some $\eta \neq \eta_0$ $M^+ = M^+(\eta) > 0$. Since M^+ is the maximal solution of (4.39), $(M^+)^{-1}$ is the minimal positive definite solution of (4.37) (with $D = I$ of course). Thus $K^+ = (M^+)^{-1}$. We can rewrite (4.39) as

$$(- A' - C'CM^+) = (M^+)^{-1}(A - BB'K^+ + 1/\eta\ K^+)M^+$$

which shows, in view of $\text{Re}\ \lambda(- A' - C'CM^+) < 0$, that

---

*<u>Proof.</u> The (FDI) corresponding to η_0 is

$$I + C(-j\omega I + A')^{-1}BB'(j\omega I + A)^{-1}C' - 1/\eta_0\ C(-j\omega I + A)^{-1}(j\omega I + A')^{-1} \geq 0$$

for all ω. We can rewrite the above (FDI) as

$$I + C(-j\omega I + A')^{-1}BB'(j\omega I + A)^{-1}C' - 1/\eta_1\ C(-j\omega I + A)^{-1}(j\omega I + A')^{-1}$$
$$\geq (1/\eta_0 - 1/\eta_1)\ C(-j\omega I + A)^{-1}(j\omega I + A)^{-1}$$

Thus at η_1 the strict form of the (FDI) holds and the result follows by Theorem A.1.2.

$$\operatorname{Re} \lambda(A - BB'K^{+} + 1/\eta\, K^{+}) \stackrel{\text{def}}{=} \operatorname{Re} \lambda(A^{+}) < 0$$

for $\eta \neq \eta_0$ i.e. that K^{+} is strictly feedback stabilizing for almost all the allowed range of η.

The end result of this discussion is the reduction of the problem of the finiteness of the inf-sup (4.34) to the existence of positive definite real solutions to

$$- AM - MA' + (BB' - 1/\eta\, I) = MC'CM \tag{4.40}$$

a question that can be answered by an iterative procedure for a given η. The determination of all η's for which $M^{+} > 0$ is straightforward since we know that the η's lie in an interval (η_0, ∞) or $[\eta_0, \infty)$. An arbitrarily small interval $[a, b]$ containing η_0 can be found by a half-interval search algorithm similar to Algorithm 1.

Algorithm 2

Step 1. Let $a = 0$, $b = M$, M arbitrary

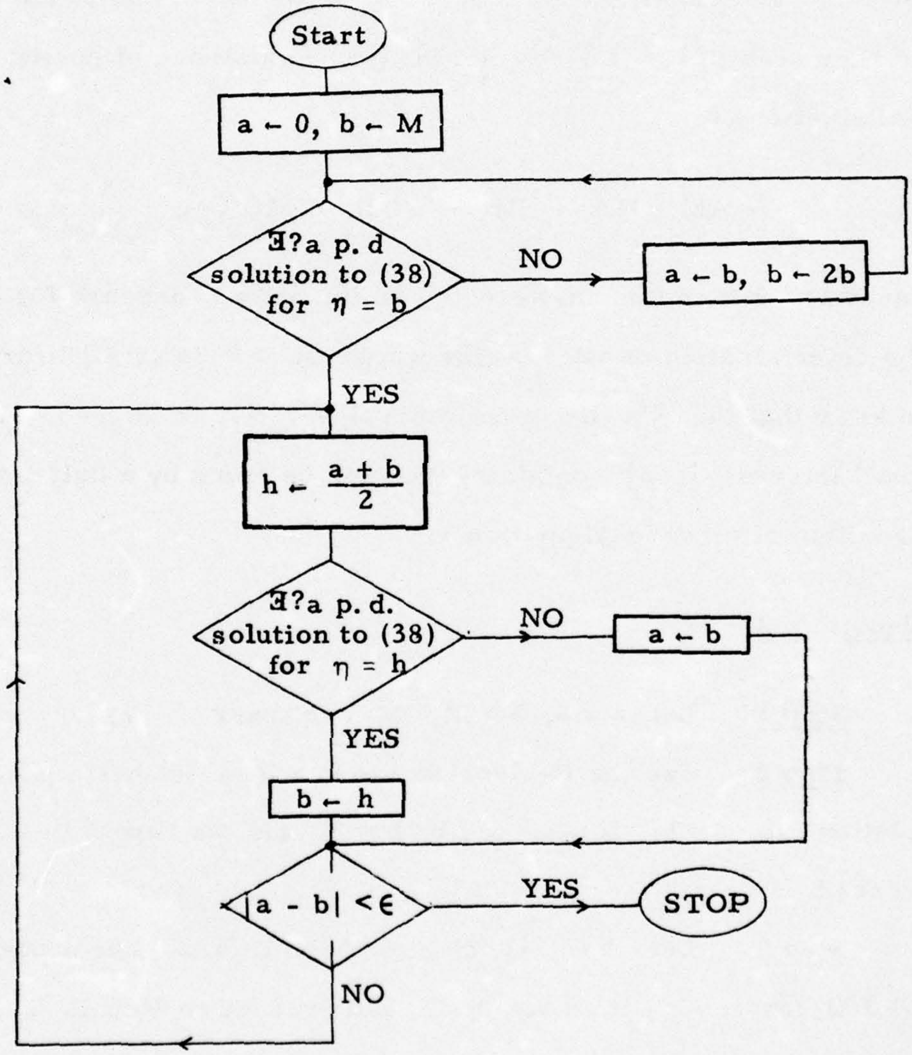
Step 2. Use the B-algorithm to see if (4.38) has a positive definite solution for $\eta = b$. If not, $a \leftarrow b$, $b \leftarrow 2b$ and we repeat this step. Else, proceed.

Step 3. Let $h \leftarrow (a+b/2)$ and check if (4.38) has a positive definite solution for $\eta = h$. If so set $b \leftarrow h$ and proceed to Step 4. Else, set $a \leftarrow h$ and proceed to Step 4.

Step 4. If $|a - b| < \epsilon$, a desired accuracy, stop. Else repeat Step 3.

Figure 3

Flowchart for Algorithm 2



The crucial step in both Algorithms 1 and 2 is the iterative solution of the (ARE) for a sequence of ρ or η's. For an implementation on a Wang 2200 computer the B-algorithm for a 10-dimensional system converges to 4 significant digits in about 10 iterations, each involving the solution of a linear matrix equation. These figures are consistent with those reported in [8]. The convergence is quadratic if the initial control happens to be close to the actual solution [8], and in practice 2-4 iterations would suffice for 4 figure accuracy if a good initial control is chosen. This happens to be the case in Algorithms 1 and 2 where the control $u = - CM^{+}(\eta)x$ would provide a good initial choice for solving (4.40) for any η_1 close to η.

The numerical bottleneck is in solving the linear matrix equation $A'K + KA = - Q$. This is equivalent to a $m = n(n+1)/2$ dimensional linear system where n is the dimension of the state. To solve this system $M = m^3/3$ multiplications are needed. For ten such iterations the computer time denoted to multiplications is approximately

$$T \doteq \frac{n^6}{24} \cdot \mu$$

μ being the machine multiplication time. For $n = 10$ and a fast machine, $\mu = 20 \times 10^{-6}$ the computer time is $T \simeq 1$ secs., an acceptable time figure. In view of the n^6-dependence of T on n, the time requirements would be prohibitive beyond $n = 30$, in which case computer time of almost an hour will be needed. Still, for system up to $n = 20$ the algorithm can be carried out effortlessly in a medium capacity machine.

3.4 The Determination of $\mathscr{R}$

The set $\mathscr{R}$, introduced in Section 3.2 consists of ρ's for which the system

$$\dot{x} = Ax + Bu + v \tag{4.41}$$

can be made dissipative in $w = v'v - \rho x'x$ following the application of a linear feedback $u = Px$. If $\rho \in \mathscr{R}$, the system

$$\dot{x} = Ax + Bu + h(x,t)$$

can be stabilized for $h'h \leq \rho\, x'x$. The set $\mathscr{R}$ provides thus a measure of our capacity to stabilize a nonlinear system. We apply here the inf-sup approach to determine whether a *given* ρ *belongs to it*.

Let us consider the problem

$$\left.\begin{aligned} J(x_0) &= \inf_u \sup_v \int_0^\infty \rho\, x'x + \theta\, u'u - v'v \quad dt \\ \dot{x} &= Ax + Bu + v \qquad x(0) = x_0 \end{aligned}\right\} \tag{4.42}$$

We showed in 3.2 that $J(x_0) < \infty$ for some ρ, θ implies that (4.41) can be made dissipative in $w = v'v - \rho\, x'x$. Now, for θ small the functional in (4.42) differs by little from the functional used in defining $S_a(x_0)$ (4.8). We expect therefore that for small enough θ the finiteness of $S_a(x_0)$ and $J(x_0)$ in (4.42) are equivalent. This is proven next in

LEMMA 4.1 For any ρ less than $\rho_0 = \text{l.u.b. } \mathscr{R}$ there exists a $\theta(\rho)$ such that the inf-sup $J(\rho, \theta, x_0)$ of (4.42) is finite for any x_0 and any θ, $0 \leq \theta \leq \theta(\rho)$.

<u>Proof.</u> Let $\rho_1 = \rho + \rho_0/2 \in \mathscr{R}$. There exists a $P = P(\rho_1)$ such that the system

$$\dot{x} = (A + BP)x + v \tag{4.43}$$

is dissipative in $w = v'v - \rho_1 x'x$. Let $m > 0$ be such that $P'P < mI$. For $J(\rho, \theta, x_0) = J(x_0)$ in (4.42) we have

$$\left.\begin{aligned} J(x_0) &\le \sup_v \int_0^\infty \rho x'x + \theta x'P'Px - v'v \quad dt \\ \dot{x} &= (A + BP)x + v \quad . \end{aligned}\right\} \tag{4.44}$$

Now $\rho I + \theta P'P \le (\rho + \theta m)I$, and for $\theta < (\rho_1 - \rho)/m = \theta(\rho)$ we have $\rho I + \theta P'P \le \rho_1 I$. Hence

$$\left.\begin{aligned} J(x_0) &\le \sup_v \int_0^\infty \rho_1 x'x - v'v \quad dt \\ \dot{x} &= (A + BP)x + v \end{aligned}\right\}$$

which is finite by our assumptions on ρ_1 and P. Q. E. D.

In view of Lemma 4.1 and the observations preceding it, we have proved

PROPOSITION 4.3: The inequality $\rho < \rho_0$ holds if and only if the inf-sup $J(x_0)$ in (4.42) is finite for some $\theta > 0$. This is equivalent to the (ARG)

$$A'K + KA + \rho I = K(\theta^{-1}BB' - I)K \tag{4.45}$$

possessing a positive definite solution for some $\theta > 0$.

Algorithm 2 of 3.3 can be used to estimate, for a given θ, the maximal $\rho = \rho(\theta)$ for which (4.45) has a positive definite solution. Successive applications of the algorithm for $\theta \to 0$ will provide a sequence $\rho(\theta) \to \rho_0$.

EXAMPLE. For the 2-dimensional example in 4.1 we showed that $\rho_0 \leq 1$. By using Proposition 4.3 we can show that $\rho_0 = 1$. Consider the inf-sup problem (4.42) where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad .$$

An elementary algebra solution of (4.45) for K gives

$$k_{11}^2 = \frac{(1-\rho)a^2}{(\rho - (1-\rho)a)^2} - 1$$

$$k_{21} = k_{12} = ((1-\rho)a - \rho)^{-1}$$

$$k_{22}^2 = k_{11}^2 \, a^{-2}$$

where

$$a^2 = \frac{\rho(1-\theta)}{(1-\rho)\theta} \quad .$$

A further computation shows that $K > 0$ if and only if $1 > \rho + \theta$, and hence

$$\rho_0 = \lim_{\theta \to 0} \rho(\theta) = \lim_{\theta \to \infty} (1-\theta) = 1 \quad .$$

3.5 Stabilization Through Observers

In this section we extend the previous methods to the decentralized stabilization of interconnected systems for which the information available to the subsystem controllers is $y_i = c_i x_i$ rather than the entire subsystem state x_i. In this case the subsystem control will be carried out by the addition of subsystem state observers [7]. We consider again the subsystems

$$\left.\begin{aligned} \dot{x}_i &= A_i x_i + B_i u_i + v_i \\ y_i &= C_i x_i \end{aligned}\right\} \tag{4.46}$$

for $i = 1, \ldots, N$. When the subsystems in (4.46) are interconnected we will have $v_i = h_i(x_1, \ldots, x_N; t)$. For a suitable control scheme on each subsystem, we will show that a condition of the form

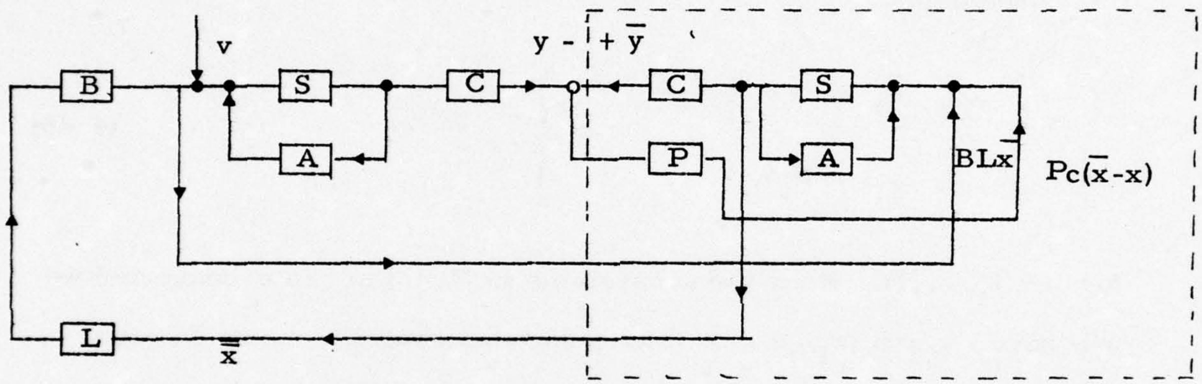
$$\sum_{i=1}^{N} h_i' h_i \leq \sum_{i=1}^{N} \rho_i \, x_i' x_i \tag{4.47}$$

will again guarantee the stability of the interconnected system. To do this we will consider the dissipativeness of the system in (4.46) in a suitable supply rate w. For simplicity we drop the subscript i and consider the system

$$\left.\begin{aligned} \dot{x} &= Ax + Bu + v \\ y &= Cx \end{aligned}\right\} \tag{4.48}$$

where (A, B, C) is controllable and observable. If $v = 0$, (4.48) can be

stabilized by first building an observer to track x by a vector $\bar{x}$ of the same dimension as x * and then letting $u = L\bar{x}$. The assumption of controllability and observability guarantee that the poles of the new system can be placed arbitrarily [7] and thus the system is stabilizable. In assessing the dissipativeness of the system, v is not null and some further analysis is needed. A diagram for an observer-stabilizer could be as follows



The design parameters here are the matrices P, L.

The equations for this system are then

$$\dot{x} = Ax + BL\bar{x} + v$$

$$\dot{\bar{x}} = A\bar{x} + BL\bar{x} + PC(\bar{x} - x)$$

Setting $e = \bar{x} - x$ we have

$$\begin{pmatrix} \dot{e} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} A+PC & 0 \\ BK & A+BL \end{pmatrix} \begin{pmatrix} e \\ x \end{pmatrix} + \begin{pmatrix} -I \\ I \end{pmatrix} v \qquad (4.49)$$

---
*We assume a non-minimal order observer.

or

$$\dot{\overline{x}} = \overline{A}\,\overline{x} + \overline{B}v, \quad \overline{x}' = (e', x'), \quad \overline{A} = \begin{pmatrix} A + PC & 0 \\ BL & A + BL \end{pmatrix}, \quad \overline{B} = \begin{pmatrix} -I \\ I \end{pmatrix}$$

Let us assume that an observer system such as (4.49) has been implemented for each subsystem in (4.46) in the parameters P_i, L_i. Furthermore, each of the subsystems

$$\begin{pmatrix} \dot{e}_i \\ \dot{x}_i \end{pmatrix} = \begin{pmatrix} A_i + P_iC_i & 0 \\ B_iL_i & A_i + B_iL_i \end{pmatrix} \begin{pmatrix} e_i \\ x_i \end{pmatrix} + \begin{pmatrix} -I \\ I \end{pmatrix} v_i \qquad (4.50)$$

is assumed dissipative in the rate

$$w(v_i, \overline{x}_i) = v_i'v_i - \rho_i \overline{x}_i (0 \;\; I) \begin{pmatrix} 0 \\ I \end{pmatrix} \overline{x}_i$$

$$= v_i'v_i - \rho_i x_i'x_i$$

The behavior of the interconnected system after the implementation of the observers is described by (4.50) for $i = 1, \ldots, N$ and $v_i = h_i(x, \ldots, x_N; t)$. In view of our previous results, the entire system will be dissipative in the rate $w = \Sigma w_i$ and stable provided

$$\sum_{i=1}^{N} h_i'(x_1, \ldots, x_N; t)\, h_i(x_1, \ldots, x_N; t) \le \sum_{i=1}^{N} \rho_i x_i'x_i$$

This is a stability condition in the desired form (4.47).

The dissipativeness of the systems in (4.50), or for simplicity that in (4.49), can be verified by using the methods developed so far. In

particular, (4.49) will be dissipative in the rate $w = v'v - \rho x'x$ provided

(a) $\bar{A}$ is stable, i.e. $A + PC$, $A + BL$ are stable

(b) The (ARE)

$$\bar{A}K + K\bar{A} - \rho \begin{pmatrix} 0 \\ I \end{pmatrix} (I \;\; 0) = K \begin{pmatrix} -I \\ I \end{pmatrix} (-I, \; I) K \tag{4.51}$$

has a negative definite solution.

For given P, L, ρ both (a) and (b) can be checked; the B-algorithm will be used for (b). The (ARE) (4.51) will have a solution for $\rho = 0$ and by continuity, for all $\rho \in [0, \rho_0]$. The upper bound ρ_0 can be approximated to a desired accuracy by using Algorithm 1.

In the case where A_i is not stable for some $i = 1, \ldots, N$, one can choose P_i, L_i such that (4.50) is stable [7]. This will guarantee that (4.50) will be dissipative for some $\rho > 0$. Indeed, the (ARE) (4.51) will possess a negative definite solution for $\rho = 0$ provided $\bar{A}$ is stable, and by continuity this property will persist for $\rho \in [0, \rho_0]$ for some $\rho_0 > 0$. We can always obtain a stability bound (4.47) albeit with ρ_i small.

The question of designing P_i, L_i in an optimal fashion, i.e. to guarantee as large a ρ_i as possible is at least as complicated as the analogous problem of Section 3.1. The design of P, L should be done simultaneously in view of their interdependence as evidenced in the system matrix $\bar{A}_i$. Such a task seems prohibitive at the moment. A reasonable suboptimal method would decouple the design by first choosing a L_i, on the basis of the game theoretic design in Sections 3.2-4, and then selecting an observer matrix P such that $A + PC$ is stable. The maximal value of ρ_i can then be determined by an application of Algorithm 1 for the matrices P, and $L = - B'K^+$.

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V. VALUES AND STRATEGIES FOR LINEAR QUADRATIC GAMES OF INFINITE DURATION

1. Introduction and Preliminaries

In the last section we introduced a zero sum differential game with value functional

$$J(u, v; x_0) = \int_0^\infty x'C'Cx + u'u - \eta\, v'v \quad dt \tag{5.1}$$

and dynamics

$$\dot{x} = Ax + Bu + Dv \qquad x(0) = x_0 \tag{5.2}$$

Our main interest was in finding conditions for the problem $\inf_u \sup_v J(u, v; x_0)$ to be finite for all x_0. We claimed that the finiteness of the inf-sup is equivalent to the stronger statement

$$V(x_0) = \inf_u \sup_v J(u, v; x_0) = \sup_v \inf_u J(u, v; x_0) < \infty \tag{5.3}$$

i.e., that a finite <u>value</u> $V(x_0)$ exists for the game (5.1). In this section we verify these claims and examine the question of values and strategies for the infinite duration game (5.1), a problem interesting in its own right. For simplicity we consider the parameter η in (5.1) to be unity without any loss of generality as having $\eta \neq 1$ is equivalent to changing the matrix D in (5.2).

Perhaps the most appealing way to show the existence of the value is to exhibit an <u>equilibrium pair</u> of strategies. This is a pair u^0, v^0 such that

$$J(u^0, v) \le J(u^0, v^0) \le J(u, v^0) \tag{5.4}$$

for all $u, v \in U, V$ the set of admissible strategies. We assume throughout this section that the admissible strategies are continuous functions $u(x,t)$, $v(x,t)$ in x, t only. *It follows from* (5.4)* *that the value of the game is* $J(u^0, v^0)$.

For a finite duration differential game, i.e., one with value functional

$$J_T(u, v; x_0) = \int_0^T x'C'Cx + u'u - v'v \quad dt \tag{5.5}$$

and dynamics as in (5.2), a pair of equilibrium strategies can be constructed by "completing the square." We need first

LEMMA 5.1. Consider the value functional (5.5) and the dynamics (5.2).

(i) Let us assume that the Riccati Equation (RE)

$$\dot{K} + A'K + KA + C'C = K(BB' - DD')K \tag{5.6}$$

with the end-point boundary condition $K(T) = 0$ has a solution on $[0, T]$ denoted by $K(t, T)$. Then for any two controls u_t, v_t $t \in [0, T]$ the value functional (5.5) subject to (5.2) becomes

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*This follows from the inequalities

$$J(u^0, v^0) = \sup_v J(u^0, v) \ge \inf_u \sup_v J(u, v) \ge \sup_v \inf_u J(u, v) \ge \inf_u J(u, v^0)$$
$$= J(u^0, v^0)$$

$$J_T(u, v; x_0) = x_0' K(0, T) x_0 + \int_0^T \| u_t + B'K(t, T) x_t \|^2 \, dt$$

$$- \int_0^T \| v_t - D'K(t, T) x_t \|^2 \, dt \qquad (5.7)$$

(ii) Let K be a real symmetric solution of the Algebraic Riccati Game (ARG) equation

$$A'K + KA + C'C = K(BB' - DD')K \qquad (5.8)$$

For any u_t, v_t $t \in [0, T]$ the functional (5.5) becomes

$$J_T(u, v; x_0) = x_0' K x_0 - x_T' K x_T + \int_0^T \left\{ \| u_t + B'Kx_t \|^2 - \| v_t - D'Kx_t \|^2 \right\} dt \qquad (5.9)$$

Expression (5.9) will be very useful for the infinite duration game. Expression (5.7) gives us an equilibrium pair of strategies for game (5.5).

PROPOSITION 5.1.* Under the assumptions of Lemma 5.1(i) the strategies

$$\left. \begin{aligned} u_T^0(x_t) &= -B'K(t, T)x_t \\ v_T^0(x_t) &= D'K(t, T)x_t \end{aligned} \right\} \qquad (5.10)$$

are in equilibrium and the value of the game is $V_T(x_0) = x_0' K(0, T) x_0$.

*Lemma 1 and Proposition 1 appear in [1].

Proof. Definition (5.4) of equilibrium pairs is immediately verified: Using (5.7) we get

$$J(u_T^0, v; x_0) = x_0' K(0, T) x_0 - \int_0^T \| v_t - D' K(t, T) x_t \|^2 \, dt$$

and thus v_T^0 is the optimal strategy for v, verifying the left inequality in (5.4). The right inequality is proved similarly. Q.E.D.

Consider now the situation where $\lim_{T \to \infty} K(0, T) = K$ exists. Then $\lim_{T \to \infty} V_T(x_0) = x_0' K x_0$. It should be expected that the value of the infinite duration is $V(x_0) = x_0' K x_0$ and this is indeed proven in Section 5.3. Thus the existence of $\lim_{T \to \infty} K(0, T)$ will guarantee a finite value for game (5.1) and hence a finite $\inf_u \sup_v J(u, v; x_0)$. Conversely, if $\lim_{T \to \infty} K(0, T)$ does not exist, it will also be shown in Section 5.3 that $\inf_u \sup_v J(u, v; x_0)$ is not finite for all x_0 and hence the finiteness of the value of (5.1) is equivalent to the finiteness of the inf-sup $J(u, v; x_0)$. In the next Section 5.2 we develop some results about the Riccati type differential equation (5.6) that will be needed in 5.3.

For a more game theoretic oriented presentation of the results in this Section see [5].

2. The Game-Riccati Equations

The state x enters the value functional (5.1) through a positive semidefinite matrix $C'C$. This fact guarantees a simple behavior of the solutions of the (ARG) (5.6). We have

LEMMA 5.2. Let $K(t, T)$, the solution of

$$\dot{K} + A'K + KA + C'C = K(BB' - DD')K \tag{5.6}$$

with endpoint condition $K(T, T) = 0$, exist for $t \in [0, T]$. Then $K(t, T)$ is nonincreasing in t, in the sense of positive matrices.

Proof. We give a game theoretic argument. In view of the time invariance of (5.6) it suffices to show that $K(0, T) \leq K(0; T+\Delta)$ for any $\Delta \geq 0$ such that $K(0, T+\Delta)$ exists. The value of the game (5.5) of duration $T+\Delta$ is (Prop. 5.1)

$$V_{T+\Delta}(x_0) = x_0' K(0, T+\Delta)x_0 \tag{5.11}$$

Consider a closed loop strategy for the maximizer in this game defined by

$$\tilde{v}(x_t) = \begin{cases} D'K(t, T)x_t & t \in [0, T) \\ 0 & t \in [T, T+\Delta] \end{cases} .$$

Note that for any u, x_0 we have (Lemma 5.1)

$$\begin{aligned} J_{T+\Delta}(u, \tilde{v}; x_0) &= x_0' K(0, T)x_0 + \int_0^T \| u_s + B'K(s, T)x_s \|^2 \, ds \\ &\quad + \int_T^{T+\Delta} \{x_s' C'Cx_s + u_s' u_s\} \, ds \geq x_0' K(0, T)x_0 . \end{aligned}$$

However, from the definition of value and (5.11) we obtain

$$\begin{aligned} x_0' K(0, T+\Delta)x_0 = V_{T+\Delta}(x_0) &\geq \inf_u J(u, \tilde{v}; x_0) \\ &\geq x_0' K(0, T)x_0 \end{aligned}$$

which proves the lemma as x_0 is arbitrary. Q.E.D.

Let us assume now that $K(0,T)$ exists for all $T > 0$ and in fact $\lim_{T\to\infty} K(0,T) = K^+$. In view of the time invariance of (5.6), this is equivalent to saying that $K(t,T)$ exists for all $t \le T$ and $\lim_{t\to-\infty} K(t,T) = K^+$. Now $\dot{K}(0,T)$ is related to $K(0,T)$ through (5.6) and thus the existence of $\lim_{T\to\infty} K(0,T)$ implies that $\lim_{T\to\infty} \dot{K}(0,T) = 0$. Therefore $K^+ = \lim_{T\to\infty} K(0,T)$ satisfies the (ARG)

$$A'K + KA + C'C = K(BB' - DD')K \tag{5.8}$$

Furthermore $K(t,T) \ge 0$ by Lemma 5.2 and thus $K(0,T) \ge 0$ and $K^+ \ge 0$. Thus K^+ is a real symmetric positive semidefinite solution of (5.8). The following lemma shows that K^+ is in fact positive definite.

LEMMA 5.3: Any real symmetric solution K of (5.8) is invertible, assuming (A, C) is observable.

Proof. Let there be a $x \neq 0$ such that $Kx = K'x = 0$. Post- and premultiplying (5.8) by x and x' we get that $x'C'Cx = 0$ and thus $Cx = 0$. By just postmultiplying (5.8) by x we get

$$A'Kx + KAx + C'Cx = K(BB' - DD')Kx$$

and hence $KAx = 0$. If $Ax = 0$, we have $Ax = 0$, $Cx = 0$ contradicting the observability assumption. If $Ax \neq 0$, by repeating the above arguments with Ax in place of x we obtain $CAx = 0$, $CA^2x, \ldots, CA^nx = 0, \ldots,$ which again contradicts the observability assumption unless $x = 0$. Q.E.D.

We showed therefore that if $\lim_{T\to\infty} K(0, T)$ exists, there exists a positive definite solution to (5.8). The converse is also true. To prove this we first provide a bound for $K(t, T)$.

LEMMA 5.4: Assume that $K(t, T)$ exists for $t \in [0, T]$. Let $K = K' > 0$ be a real solution to (5.8). Then $K(t, T) \le K$ *for* $t \in [0, T]$.

<u>Proof.</u> We give a game-theoretic proof. Let us assume that the minimizer uses the strategy $\tilde{u} = -B'Kx$. The value functional for a game (5.5) of duration $T_1 \le T$ can be written for any v as

$$J_{T_1}(\tilde{u}, v; x_0) = x_0' K x_0 - x_{T_1}' K x_{T_1} - \int_0^{T_1} \| v_t - D'Kx_t \|^2 \, dt \tag{5.12}$$

Use of Lemma 5.1(ii) was made in deriving (5.12). It follows from the assumption $K > 0$ that for all v, $J_{T_1}(\tilde{u}, v; x_0) \le x_0' K x_0$ and hence

$$x_0' K x_0 \ge \sup_v J_{T_1}(\tilde{u}, v; x_0) \ge \inf_u \sup_v J_{T_1}(u, v; x_0) \; . \tag{5.13}$$

The value of game (5.5) is thus less than $x_0' K x_0$ and we conclude from (5.13) that

$$V_T(x_0) = x_0' K(0, T_1) x_0 \le x_0' K x_0$$

and $K \ge K(0, T_1) \ge K(t, T_1)$ for $0 \le t \le T_1$ and $T_1 \le T$. Q.E.D.

PROPOSITION 5.2: Under the assumption of observability, $\lim_{T\to\infty} K(0, T)$ exists iff there exists a positive definite solution to (5.8).

<u>Proof.</u> If $K = K' > 0$ is a solution to (5.8), then $K(t, T)$ exists for any T and any $t \in [0, T]$. This follows from the fact that $K(t, T)$ is nondecreasing as t decreases and can fail to exist only if it increases without a bound. This cannot happen in view of Lemma 5.4. Thus $K(0, T)$ exists for all T, and in fact it is *increasing in* T and bounded by K. Thus $\lim_{T\to\infty} K(0, T)$ exists. The converse has been proven already.

Q. E. D.

Proposition 5.2 implies that $K^+ = \lim_{T\to\infty} K(0, T)$ is the smallest positive definite solution of (5.8). Indeed, if $K = K' > 0$ is a solution of (5.8), $K \geq K(0, T)$ and hence $K \geq \lim_{T\to\infty} K(0, T) = K^+$. This observation is used to give an important property of K^+.

LEMMA 5.5: K^+ is feedback stabilizing, i.e. $\mathrm{Re}\,\lambda(A - BB'K^+ + DD'K^+) \leq 0$.

<u>Proof.</u> Note that $(K^+)^{-1} \geq 0$ exists and satisfies the (ARG)

$$-AM - MA' + (BB' - DD') = MC'CM \tag{5.14}$$

According to Theorem A. 1. 1 of Appendix 1, there exists a maximal solution M^+ of (5.14), i.e., such that $M^+ \geq (K^+)^{-1} > 0$. Furthermore M^+ satisfies $\mathrm{Re}\,\lambda(-A' - C'CM^+) \leq 0$. Now $0 < (M^+)^{-1} \leq K^+$ and $(M^+)^{-1}$ is a solution of (5.8), contradicting the minimality of K^+ among the positive definite solutions of (5.8) unless $(M^+)^{-1} = K^+$. By rearranging (5.14) we obtain that

$$(-A' - C'CM^+) = K^+(A - BB'K^+ + DD'K^+)(K^+)^{-1}$$

and hence

$$\mathrm{Re}\,\lambda(A - BB'K^{+} + DD'K^{+}) = \mathrm{Re}\,\lambda(-A' - C'CM^{+}) \leq 0 \quad .$$

Q. E. D.

3. The Value of the Infinite Duration Game

In this section we assume that there exists a solution $K = K' > 0$ to (5.8). It follows from Proposition 5.2 that $\lim_{T\to\infty} K(0, T) = K^{+}$. In view of the time invariance of (5.6) we also have $\lim_{T\to+\infty} K(t, T) = K^{+}$ for any fixed t. In order to show that $V(x_0) = x_0' K^{+} x_0$ is the value of game (5.1) it would suffice to exhibit an equilibrium pair of strategies, as was done in Proposition 5.1 for finite duration games. In fact the limit of the equilibrium strategies for the finite duration game $u_T^0(x_t)$, $v_T^0(x_t)$ as $T \to \infty$ become

$$\left.\begin{aligned} u^0(x_t) &= -B'K^{+}x_t \\ v^0(x_t) &= D'K^{+}x_t \end{aligned}\right\} \tag{5.15}$$

and are obvious candidates for being in equilibrium. It is mildly surprising that u^0, v^0 need not be in equilibrium. In fact, consider a scalar game with value functional

$$J(u, v; x_0) = \int_0^\infty x^2 + u^2 - v^2 \ dt$$

and dynamics

$$\dot{x} = x + u + \frac{1}{\sqrt{2}} v \qquad x(0) = x_0$$

The Riccati differential equation is

$$\dot{k} = \frac{1}{2}k^2 - 2k - 1 \qquad k(T) = 0$$

which can be integrated to give

$$k(t,T) = \sqrt{6}\tanh[\sqrt{6}(T-t) + \tanh^{-1}(-2/\sqrt{6})] + 2$$

Now $\lim_{T\to\infty} k(t,T) = 2+\sqrt{6}$ and the strategy pair in (5.15) is $u^0(x_t) = -(2+\sqrt{6})x_t$ and $v^0(x_t) = (2+\sqrt{6}/\sqrt{2})x_t$. These strategies are not in equilibrium. If the maximizer uses the strategy $v^0(x_t)$, the minimizer is faced with the least squares problem

$$\operatorname*{Inf}_{u} \int_0^\infty x^2 + u^2 - (v^0(x))^2 \; dt = \int_0^\infty [(-4 - 2\sqrt{6})x^2 + u^2] \; dt$$

and dynamics

$$\dot{x} = x + u + \frac{1}{\sqrt{2}} v^0(x)$$

$$= \frac{4+\sqrt{6}}{2} x + u \qquad x(0) = x_0$$

The minimizer's optimal response is not $u^0(x_t)$. In fact, by playing $u(x_t) = 0$ the value functional becomes $J(0, v^0(x); x_0) = -\infty$ for $x_0 \neq 0$. Thus the right hand inequality in the definition of equilibrium strategies (5.4) is not satisfied.

We next prove that $x_0' K^+ x_0$ is nevertheless the value of game (5.1) by invoking the definition of value (5.3) directly instead of exhibiting an equilibrium pair. Such a pair need not even exist for this game.

PROPOSITION 5.3: Let (A, C) be observable and let $K = K' > 0$ be a solution of (5.8). The value of game (5.1) is $V(x_0) = x_0' K^+ x_0$.

Proof. We have to show according to (5.3) that

$$\inf_u \sup_v J(u, v; x_0) = \sup_v \inf_u J(u, v; x_0) = x_0' K^+ x_0$$

where the inf, sup are taken over the closed loop strategies. This is equivalent to showing that for any $\varepsilon > 0$ there exist strategies $u_\varepsilon(x)$, $v_\varepsilon(x)$ such that

$$J(u_\varepsilon, v; x_0) \leq x_0' K^+ x_0 + \varepsilon \qquad \forall v \tag{5.16}$$

$$J(u, v_\varepsilon; x_0) \geq x_0' K^+ x_0 - \varepsilon \qquad \forall u \tag{5.17}$$

We know that $\lim_{T \to \infty} K(0, T) = K^+$ exists. Consider the strategy $u^0(x_t) = -B'K^+ x_t$. We will show that it satisfies (5.16) for any ε. In fact, consider

$$\sup_v J(u^0, v; x_0) = -\inf_v \int_0^\infty \left\{ -x'(C'C + K^+ BB'K^+)x + v'v \right\} dt$$

$$\dot{x} = (A - BB'K^+)x + Dv \qquad x(0) = x_0$$

The (ARE) for this problem is

$$(A - BB'K^+)'M + M(A - BB'K^+) - C'C - K^+ BB'K^+ = MDD'M \tag{5.18}$$

Note that (5.18) holds for $M = -K^+ < 0$ and in addition $-K^+$ is feedback stabilizing for this problem, i.e.

$$\text{Re}\,\lambda(A-BB'K^{+} - (-DD'K^{+})) = \text{Re}\,\lambda(A-BB'K^{+} + DD'K^{+}) \leq 0$$

according to Lemma 5.5. It follows from Theorem A.1.3 that the infimum is $-x_0'K^{+}x_0$ or $\sup_v J(u^0, v; x_0) = x_0'K^{+}x_0$ which proves (5.16) with $\epsilon = 0$.

To construct a $v_\epsilon(x_t)$ satisfying (5.17) for a given ϵ and x_0 we work as follows. There exists a $T = T(\epsilon, x_0)$ such that

$$x_0'K^{+}x_0 - \epsilon \leq x_0'K(0;T)x_0 \tag{5.19}$$

This follows from $\lim_{T\to\infty} K(0,T) = K^{+}$. Consider now the strategy

$$v_\epsilon(t) = \begin{cases} D'K(t;T)x_t & t \in [0,T) \\ 0 & [T,\infty) \end{cases} \tag{5.20}$$

This corresponds to playing optimally for a game of duration T for $t \in [0,T)$ and using no control from then on. We denote the integrand of the functional (5.1) as $w(u,v,x)$ and split the integral as

$$J(u, v_\epsilon; x_0) = \int_0^\infty w(u, v_\epsilon, x)\ dt = \int_0^T w(u, v_\epsilon, x)\ dt + \int_T^\infty w(u, v_\epsilon, x)\ dt \tag{5.21}$$

The last integral in (5.21) is nonnegative by the definition of $v_\epsilon(x)$, i.e., it equals $\int_T^\infty [x'C'Cx + u'u]\ dt$.

We can express the *first* integral according to Lemma 5.1(i) as

$$\int_0^T w(u, v_\varepsilon, x)\ dt = x_0' K(0,T) x_0 + \int_0^T \| u_t + B'K(t,T)x_t \|^2\ dt$$

$$- \int_0^T \| v_\varepsilon - D'K(t,T)x_t \|^2\ dt \qquad (5.22)$$

The last integral vanishes by the definition of $v_\varepsilon(t)$ and thus

$$\int_0^T w(u, v_\varepsilon, x)\ dt \geq x_0' K(0,T) x_0 .$$

Hence $J(u, v_\varepsilon; x_0) \geq x_0' K(0,T) x_0$. Using (5.19) we finally obtain

$$J(u, v_\varepsilon; x_0) \geq x_0' K(0,T) x_0 \geq x_0' K^+ x_0 - \varepsilon$$

which is exactly (5.17). Q.E.D.

A converse to Proposition 5.3 would state that game (5.1) does not have a value if the (ARG) (5.8) does not have a positive definite solution. The nonexistence of a positive definite solution of (5.8) implies that $\lim_{T\to\infty} K(0,T)$ does not exist. Since $K(0,T)$ is nondecreasing in T it follows that $x_0' K(0,T) x_0$ increases without bound in T (for some $x_0 \neq 0$), i.e., for any $M > 0$, there exist a T such that $x_0' K(0,T) x_0 > M$. We can use this fact to show that $\inf_u \sup_v J(u,v;x_0) \geq \sup_v \inf_u J(u,v;x_0) > M$. For a strategy $v_M(x)$ defined by analogy to $v_\varepsilon(x)$ in (5.20), i.e.

$$v_M(x_t) = \begin{cases} D'K(t,T)x_t & t \in [0,T) \\ 0 & t \in [T,\infty) \end{cases}$$

we showed that

$$J(u, v_M; x_0) \geq x_0' K(0,T) x_0 > M$$

Hence

$$\sup_v \inf_u J(u, v; x_0) \geq \inf_u J(u, v_M; x_0) > M$$

and since M is arbitrary it follows that both $\inf_u \sup_v$ and $\sup_v \inf_u$ are infinite. This shows that the game does not have a finite value and also shows that the finiteness of the inf-sup problem is equivalent to the finiteness of the value, as was claimed in Section 4.

Remark 1. The above results were derived on the assumption of prior commitment of the players to their respective closed loop strategies. The players' role is simply to implement mechanically a memoryless strategy whose only input is the observation of x. After the implementation, the players let the game proceed without further interference. The situation would be much different if the players were allowed to change strategies during the game, say at time t, on the basis of the entire history $\{x(\tau),\ 0 \leq \tau \leq t\}$.

The examination of games where prior commitment does not hold is equivalent to expanding the strategy sets of the players by allowing the control at time t to depend on the available memory, i.e. part of the past observations $\{x(\tau),\ \tau \in [0,t]\}$ the controls applied $\{v(\tau), \tau \in [0,t]\}$ etc. Even in this expanded formulation the existence of K^+ implies that the value of the game is $V(x_0) = x_0' K^+ x_0$, since each player can get ϵ-close to that value regardless of the control time-function employed by his opponent. An interesting problem is to find whether equilibrium strategies exist in the expanded strategy space where some kind of memory is allowed.

Remark 2. We examine in some more detail why the strategies $u^0(x) = -B'K^+x$, $v^0(x) = D'K^+x$ fail to be in equilibrium. The first half of the proof of Proposition 5.3 shows that $u^0(x)$ is a desirable strategy since $\sup_v J(u^0, v; x_0) = x_0' K^+ x_0$, i.e. $u^0(x)$ guarantees that the minimizer attains the actual value of the game. In fact, if $\operatorname{Re} \lambda(A - BB'K^+ + DD'K^+) < 0$ the proof in Proposition 5.3 can be strengthened to show that $v^0(x)$ is the optimal response to $u^0(x)$. However, the situation is not always good for the maximizer if he commits himself to $v^0(x)$. The scalar example examined earlier shows that the minimizer can take advantage of this commitment to drive J to $-\infty$. Consider the general problem faced by the minimizer when he knows that the maximizer is committed to $v^0(x) = D'K^+x$

$$\left.\begin{aligned} &\inf_u \int_0^\infty x'[C'C - K^+DD'K^+]x + u'u \\ &\dot{x} = (A + DD'K^+)x + Bu \qquad x(0) = x_0 \end{aligned}\right\} \tag{5.23}$$

The corresponding (ARG) is

$$(A + DD'K^+)'M + M(A + DD'K^+) + CC' - K^+DD'K^+ = MBB'M \tag{5.24}$$

and (5.23) will have a finite infimum iff (5.24) has a nonpositive definite solution. It is easy to check that $M = K^+$ is a solution of (5.24) and is in fact the maximal solution since K^+ is feedback stabilizing. Furthermore if (5.23) possesses a negative definite solution, the infimum will be $x_0'K^+x_0$ and will be attained by $u = -B'K^+x$. These remarks rely on Theorem A.1 of Appendix 1. Thus a necessary and sufficient condition for

$u^0(x) = -B'K^+x_t$ and $v^0(x) = D'K^+x$ to be in equilibrium is that

$$\left.\begin{array}{ll}\text{(i)} & \operatorname{Re}\lambda(A - BB'K^+ + DD'K^+) < 0 \\ \text{(ii)} & \text{There exists a negative definite solution to (5.24)}\end{array}\right\} \qquad (5.25)$$

Condition (5.25) is perhaps unsatisfactory since it involves K^+ in addition to the parameters of the game. In view of the difficulty of finding alternative conditions for a least squares problem to be finite (See [4]), it seems that (5.25) are as good conditions as one can hope for.

<u>Remark 3.</u> A reason for the failure of $v^0(x)$ is that it might pay for the minimizer to drive the state x far from the origin if, in addition, he is ensured of a negative value functional. The strategies $v_\epsilon(x)$ guard against this by stipulating that $v = 0$ for large t. The minimizer faces the positive functional $\int_T^\infty \{x'C'Cx + u'u\}\,dt$ and he will try to drive x back to the origin. In view of this remark one could conjecture that a simple set of ϵ-optimal strategies for v could be obtained by truncating $v^0(x) = D'K^+x$ for large T, i.e. using

$$v_T(x_t) = \left\{\begin{array}{ll} D'K^+x_t & t \in [0, T) \\ 0 & t \in [T, \infty) \end{array}\right\} \qquad (5.26)$$

The conjecture is that for large T, $J(u, v_T; x_0) \geq x_0'K^+x_0 - \epsilon$ for all u. Consider

$$\left.\begin{array}{l} \inf\limits_{u} J(u, v_T; x_0) = \int_0^T w(u, v^0(x), x)\, dt + \int_T^\infty x'C'Cx + u'u \quad dt \\ \dot{x} = (A + DD'K^+)x + Bu \qquad x(0) = x_0; \quad \text{for} \quad t \in [0, T) \\ \dot{x} = Ax + Bu \qquad\qquad \text{for} \quad t \in [T, \infty) \end{array}\right\} \tag{5.27}$$

For time T and on, the optimal response of the minimizer is $u^0(x) = -B'Px$ where $P = P' > 0$ is the solution of

$$A'P + PA + C'C = PBB'P$$

and

$$\inf_u \int_T^\infty \{x'C'Cx + u^0(x)' \cdot u^0(x)\}\, dt = x_T'Px_T \quad .$$

Using the expressions in Lemma 5.1 we can write the functional in (5.27) as

$$J(u, v_T; x_0) = x_0'K^+x_0 + \int_0^T \|u_t + B'K^+x_t\|^2 \quad dt + x_T(P - K^+)x_T$$

and making the transformation $u_1 = u + B'K^+x$ we finally reduce (5.27) to

$$\left.\begin{array}{l} \inf\limits_{u} \int_0^T u'u \;\, dt + x_T(P - K^+)x_T + x_0K^+x_0 \\ \dot{x} = A^+x + Bu = (A - BB'K^+ + DD'K^+)x + Bu \quad x(0) = x_0 \end{array}\right\} \tag{5.28}$$

The value of the infimum is $x_0'K(0, T)x_0$ provided a *solution* $K(t, T)$ of

$$\dot{K} + (A^+)'K + KA^+ = KBB'K \tag{5.29}$$

with the endpoint condition $K(T,T) = P - K^+$ exists on $[0,T]$.

The stabilizing solution of the (ARG)

$$(A^+)'K + KA^+ = KBB'K \tag{5.30}$$

corresponding to (5.29) is $K_1^+ = 0$ provided $\mathrm{Re}\,\lambda(A^+) \le 0$. According to Remark 21, [3] $\lim_{T\to\infty} K(0;T) = K_1^+ = 0$ provided

$$P - K^+ > K_1^- \tag{5.31}$$

where K_1^- is the minimal solution of (5.30). Thus if (5.31) holds, $\lim_{T\to\infty} J(u, v_T; x_0) \ge x_0' K^+ x_0$ regardless of u. The strategies $v_T(x)$ of (5.26) are more appealing than the $v_\epsilon(x)$, being a truncation of time-invariant linear strategies whereas $v_\epsilon(x)$ is time-varying in an intricate fashion. It should be noted though that (5.31) does not always hold and hence the $v_T(x)$ strategies are not always applicable, as for instance in the following example:

$$\left.\begin{aligned} J(u,v;x_0) &= \int_0^\infty x^2 + u^2 - v^2 \quad dt \\ \dot{x} &= x + u + \sqrt{\tfrac{2}{3}}\, v \qquad x(0) = x_0 \end{aligned}\right\} \tag{5.32}$$

The (ARG) for this problem is $2k + 1 = 1/3\, k^2$ and

$$k^+ = 3 + \sqrt{12}, \qquad A^+ = -\frac{\sqrt{12}}{3}, \qquad k_1^- = -\frac{2\sqrt{12}}{3}$$

In addition, $p > 0$ is the solution of $2p + 1 = p^2$ i.e., $p = 1 + \sqrt{2}$.

Condition (5.31) is violated since

$$(1+\sqrt{2}) - (3-\sqrt{12}) \not> -\frac{2}{3}\sqrt{12}$$

By using similar calculations as above it can be shown that

$$\sup_u J(u, v^0(x); x_0) = x_0' K^+ x_0$$

provided

$$-K^+ > K_1^- \tag{5.33}$$

The fact that $P > 0$ shows that (5.31) might hold even if (5.33) does not. The usefulness of the $v_T(x)$ strategies is evident since they can be successful even when $v^0(x)$ is not.

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APPENDIX 1

LEAST SQUARES PROBLEMS AND THE ALGEBRAIC RICCATI EQUATION

We summarize here some results in [1], [2] on the relation between the least squares problem

$$\left.\begin{aligned} V_f(x_0) &= \inf_u \int_0^\infty \{x'Qx + 2u'Px + u'u\} \; dt \\ \dot{x} &= Ax + Bu \quad x(0) = x_0 \; ; \; (A, B) \text{ controllable} \end{aligned}\right\} \tag{A. 1}$$

the algebraic Riccati equation (ARE)

$$A'K + KA + Q = (KB + P')(B'K + P) \tag{A. 2}$$

and the frequency domain inequality (FDI) for $\omega \in R$

$$\begin{aligned} H(-j\omega, j\omega) = {} & I + P(j\omega I - A)^{-1}B + B'(-j\omega I - A')^{-1}P' \\ & + B'(-j\omega I - A')Q(j\omega I - A)^{-1}B \geq 0 \end{aligned} \tag{A. 3}$$

THEOREM A. 1: (1) The (ARE) (A. 2) has a real symmetric solution $K = K'$ if and only if the (FDI) (A. 3) holds. Under this condition there will exist a unique maximal solution K^+ and a unique minimal solution K^- of (A. 2) such that for any solution $K = K'$

$$K^- \leq K \leq K^+$$

In addition, if $A^+ = A - B(B'K^+ + P)$, $A^- = A - B(B'K^- + P)$ then $\text{Re}\,\lambda(A^+) \leq 0$, $\text{Re}\,\lambda(A^-) \geq 0$.

(2) The following relations are equivalent:

(i) $H(-j\omega, j\omega) \geq \epsilon B'(-j\omega I - A')^{-1}(j\omega I - A)^{-1}B; \quad \epsilon > 0$

(ii) $\operatorname{Re} \lambda(A^+) < 0$

(iii) $\operatorname{Re} \lambda(A^-) > 0$

(iv) $K^+ - K^- > 0$.

(3) Consider the infimum $V_f(x_0)$ in (A. 1). It is finite, $V_f(x_0) > -\infty$ if and only if $K^- \leq 0$. If $K^- < 0$ $V_f(x_0) = x_0' K^+ x_0$. If $K^- < 0$, $K^- < K^+$ the infimum is attained by the stabilizing control $u = -(B'K + P)x$.

(4). Consider the least squares problem in (A. 1) with the additional constraint $\lim_{t\to\infty} x_t = 0$. Denote the value of the infimum by $V(x_0)$. Then $V(x_0) > -\infty$ if and only if the (ARE) (A. 2) has a real symmetric solution, in which case $V(x_0) = x_0' K^+ x_0$. If in addition $\operatorname{Re} \lambda(A^+) < 0$ (see 2) the infimum is attained by the stabilizing control $u = -(B'K + P)x$.

These results are proven in [2]. For the case where $P = 0$, $Q = -C'C$ we have

THEOREM A. 2 [1]: Let (A, B, C) be controllable and observable and that $\operatorname{Re} \lambda(A) < 0$. Then the (ARE)

$$A'K + KA - C'C = KBB'K \tag{A. 4}$$

has a unique negative definite solution K^+ such that $\operatorname{Re} \lambda(A^+) \leq 0$ if and only if the following (FDI) holds for all $\omega \in R$

$$I - B'(-j\omega I-A')^{-1}C'C(j\omega I-A)^{-1}B \geq 0 \tag{A. 5}$$

Theorem A. 2 plays an important role in the proof of the more general results of Theorem A. 1. It is particularly important in the development of Section IV.

APPENDIX 2

NUMERICAL RESULTS

To demonstrate the convergence of the Kleinman type algorithm under the conditions expressed in Proposition 3.58 let us consider the game Riccati equation encountered in Sections IV and V

$$A'K + KA + I = K(BB' - \eta^{-1}I)K \qquad (A.1)$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

The real symmetric solutions of (A. 1) are invertible. Setting $M = K^{-1}$ we obtain the equation

$$-AM - MA' + (BB' - \eta^{-1}I) = MM \qquad (A.2)$$

to which the Kleinman algorithm is applicable.

The following relations can be obtained by an algebraic solution of (A. 1) where $p = \eta^{-1}$

$$k_{11}^2 = k_{22}^2 = \frac{1}{p}\left(\frac{1-p}{(1-2p)^2} - 1\right) \qquad (A.3)$$

$$k_{12} = k_{21} = \frac{1}{1-2p} \qquad (A.4)$$

For a real solution to exist, the r. h. s. of (A. 3) must be positive. This happens for

$$0 \le p \le .75 \qquad (A.5)$$

However, the positive definiteness conditions, and hence $K = K' > 0$, are satisfied for

$$0 \le p < 1/2 \qquad \text{(A. 6)}$$

We present some computer results on the application of Kleinman's algorithm to (A. 2). The printout includes

(i) $P = \eta^{-1}$

(ii) The "Cost Matrix' $BB' - pI$

(iii) The "Solution Matrix", $(M_i)^{-1}$

(iv) The "Error Matrix", i. e. the l. h. s. of (A. 1) with $K = (M_i)^{-1}$.

According to Proposition 3. 58 the M_i's are decreasing in i. Hence our "solution matrices" are increasing in every iteration, being equal to $(M_i)^{-1}$.

The exact solutions $K^+(p)$, $0 \le p \le .5$, as given by (A. 3), (A. 4) are:

$$K^+(0) = \begin{pmatrix} \sqrt{3} & 1 \\ 1 & \sqrt{3} \end{pmatrix}$$

$$K^+(.45) = \begin{pmatrix} \sqrt{120} & 10 \\ 10 & \sqrt{120} \end{pmatrix} = \begin{pmatrix} 10.955 & 10 \\ 10 & 10.955 \end{pmatrix}$$

$$K^+(.498) = \begin{pmatrix} 251 & 250 \\ 250 & 251 \end{pmatrix}$$

For $0.5 \le p \le .75$ the iterates M_i are not positive definite. The algorithm still converges. The exact solution for $p = .7$ is

$$K(.7) = \begin{pmatrix} -\frac{\sqrt{5}}{2} & -2.5 \\ -2.5 & -\frac{\sqrt{5}}{2} \end{pmatrix} = \begin{pmatrix} -1.19 & -2.5 \\ -2.5 & -1.19 \end{pmatrix}$$

Finally, when p is set above .75 the numerical procedure becomes unstable. In our implementation of Kleinman's algorithm, the Lyapunov equation

$$A'K + KA = -Q$$

is solved by Davison's method (*) which essentially computes

$$K = \int_0^\infty \exp(A't)Q \exp(At) \; dt \quad .$$

If A is not a stable matrix this computation will be infeasible. In view of Proposition 3.58, the instability of $A - BB'K_i$ for some i shows that the corresponding Riccati equation does not have a strictly stabilizing solution.

(*) Davison, E. and Man, F., IEEE-AC, Vol. AC-13, p. 448.

Computer Output.

DATA

A MATRIX

```
 0  0
-1  0
```

CONTROL MATRIX

```
1  0
0  1
```

BB'-MATRIX

```
0  0
0  1
```

INITIAL CONTROL

```
2  0
0  2
```

P= 0

COST MATRIX

```
0  0
0  1
```

RESULTS

ITERATION NO. 1

SOLUTION MATRIX

```
.9275314881246   .2318817357862
.2318817357863   .8579660699744
```

A P.D. MATRIX

ERROR MATRIX

```
-.809582128518 -.498049499134
-.498049499134 -.660172226303
```

ITERATION NO. 2

SOLUTION MATRIX

```
1.497006519354   .7372599491449
.7372599949145   1.418589466237
```

A P.D. MATRIX

ERROR MATRIX

```
-9.05529655E-02 -8.64535821E-02
-8.64535821E-02 -.115317895755
```

ITERATION NO. 3

SOLUTION MATRIX

```
1.722541147259   .9875463793642
.9875463779363   1.714852474011
```

A P.D. MATRIX

ERROR MATRIX

```
-2.02742492E-03 -3.50718699E-03
-3.50718699E-03 -6.97710066E-03
```

ITERATION NO. 4

SOLUTION MATRIX

```
1.732031276368   .9999741612953
.999974161295    1.732003691797
```

A P.D. MATRIX

ERROR MATRIX

```
-3.99519000E-06 -1.66496010E-06
-1.66495990E-06 -3.39258210E-05
```

P= .45

COST MATRIX

```
-.45  0
 0   .55
```

RESULTS

ITERATION NO. 1

SOLUTION MATRIX

```
2.133322277379   1.066655911962
1.066655911961   1.823641342767
```

A P.D. MATRIX

ERROR MATRIX

```
-.2640644173277 -.2179704711008
-.2179704711008 -.200787436299
```

ITERATION NO. 2

SOLUTION MATRIX

```
6.505846880924   5.367746727958
5.367746727972   6.111888973841
```

A P.D. MATRIX

ERROR MATRIX

```
-2.14336357E-02 -2.92906686E-02
-2.92906686E-02 -4.32714941E-02
```

ITERATION NO. 3

SOLUTION MATRIX

```
10.77668070341   9.809192863188
9.809192863193   10.74842878714
```

A P.D. MATRIX

ERROR MATRIX

```
-1.99443268E-04 -5.44957521E-04
-5.44957521E-04 -2.09994619E-03
```

ITERATION NO. 4

SOLUTION MATRIX

```
10.95430638835  9.999804967345
9.999804967349  10.95419189984
```

A P.D. MATRIX

ERROR MATRIX

```
1.82273700E-06  8.35688800E-07
8.35689000E-07 -1.11999130E-05
```

ITERATION NO. 5

SOLUTION MATRIX

```
10.95443125571  9.999940172555
9.999940172543  10.95434174828
```

A P.D. MATRIX

ERROR MATRIX

```
1.90171900E-06  5.05507600E-07
5.05506800E-07 -8.00926500E-06
```

```
P= .498

COST MATRIX

-.498  0
 0   .502

          RESULTS

        ITERATION NO. 1

  SOLUTION MATRIX

 2.279190460644   1.13958964621
 1.139589646211   1.90134316303

  A P.D. MATRIX

  ERROR MATRIX

-.2805041512517 -.2337505155757
-.2337505155754 -.20300705908

        ITERATION NO. 2

  SOLUTION MATRIX

 11.29227204975   9.864318617694
 9.864318617698   10.40485104975

  A P.D. MATRIX

  ERROR MATRIX

-2.51389314E-02 -3.42460840E-02
-3.42460840E-02 -4.95421700E-02

        ITERATION NO. 3

  SOLUTION MATRIX

 148.4825933774   147.1637329674
 147.1637329675   147.8376117229

  A P.D. MATRIX
```

ERROR MATRIX

```
-2.97789232E-04 -7.88505843E-04
-7.88505842E-04 -3.11788574E-03
```

ITERATION NO. 4

SOLUTION MATRIX

```
250.770740138    249.7708493367
249.7708493361   250.7669512672
```

A P.D. MATRIX

ERROR MATRIX

```
1.95295300E-06  1.13730000E-06
1.13729880E-06 -1.54822300E-05
```

ITERATION NO. 5

SOLUTION MATRIX

```
250.9970560805  249.9978118347
249.9978118344  250.9945733825
```

A P.D. MATRIX

ERROR MATRIX

```
2.46498700E-06  6.19717000E-08
6.19711000E-08 -7.58947000E-06
```

ITERATION NO. 6

SOLUTION MATRIX

```
250.9968528565  249.9976094185
249.9976094189  250.9943717819
```

A P.D. MATRIX

ERROR MATRIX

```
2.46175000E-06  6.09831000E-08
6.09839000E-08 -7.58425300E-06
```

P= .7

COST MATRIX

```
-.7  0
 0  .3
```

RESULTS

ITERATION NO. 1

SOLUTION MATRIX

```
3.199983347867  1.599983833841
1.599983833841  2.338441689366
```

A P.D. MATRIX

ERROR MATRIX

```
-.3812507316127 -.2843768842542
-.284376884254 -.228129351652
```

ITERATION NO. 2

SOLUTION MATRIX

```
-3.512988727674 -4.233052583967
-4.233052583961 -2.585131697757
```

NOT A P.D.MATRIX

ERROR MATRIX

```
-5.70097019E-02 -6.69855371E-02
-6.69855371E-02 -8.74694405E-02
```

ITERATION NO. 3

SOLUTION MATRIX

```
-1.326607664785 -2.6143039422
-2.614303942198 -1.249297734187
```

NOT A P.D.MATRIX

ERROR MATRIX

-3.29340872E-03 -4.99885925E-03
-4.99885925E-03 -2.00415342E-02

ITERATION NO. 4

SOLUTION MATRIX

-1.126436499108 -2.505166426247
-2.505166426244 -1.126860156841

NOT A P.D.MATRIX

ERROR MATRIX

-2.79892663E-04 2.13584446E-04
 2.13584445E-04 -1.00290206E-03

ITERATION NO. 5

SOLUTION MATRIX

-1.118086249761 -2.500012449136
-2.500012449136 -1.118040270598

NOT A P.D.MATRIX

ERROR MATRIX

 2.18454600E-06 -3.34096080E-06
-3.34096070E-06 -7.54920700E-06

ITERATION NO. 6

SOLUTION MATRIX

-1.118056363739 -2.499987532182
-2.499987532183 -1.118000649691

NOT A P.D.MATRIX

ERROR MATRIX

 4.49479000E-06 -4.44695460E-06
-4.44695460E-06 -5.51742100E-06

```
P= .751

COST MATRIX

-.751  0
 0   .249

          RESULTS

     ITERATION NO. 1

  SOLUTION MATRIX

-1.629107234723 -2.737883768612
-2.737883768609 -1.218498805136

  NOT A P.D.MATRIX

  ERROR MATRIX

-5.308911161E-02 -3.890295555E-02
-3.890295555E-02 -8.520598851E-02


     ITERATION NO. 2

  SOLUTION MATRIX

-.556788572646   -2.10840422913
-2.10840422913   -.518765044692

  NOT A P.D.MATRIX

  ERROR MATRIX

-9.37791271E-03 -2.69752505E-03
-2.69752505E-03 -2.62481507E-02


     ITERATION NO. 3

  SOLUTION MATRIX

-.24413349532    -2.026846204454
-2.026846204455 -.2503422792094

  NOT A P.D.MATRIX

  ERROR MATRIX

-3.99896352E-03  8.68285065E-04
 8.68285065E-04 -5.46800553E-03
```

ITERATION NO. 4

SOLUTION MATRIX

```
-8.94844002E-02 -2.004206419456
-2.004206419457 -8.98844835E-02
```

NOT A P.D.MATRIX

ERROR MATRIX

```
-1.55104205E-03  4.77393704E-05
 4.77393704E-05 -1.44997066E-03
```

ITERATION NO. 5

SOLUTION MATRIX

```
 4.44404465E-02 -2.001001410257
-2.001001410279  4.49123144E-02
```

NOT A P.D.MATRIX

ERROR MATRIX

```
-1.12693011E-03 -5.89318143E-05
-5.89318144E-05 -1.12232356E-03
```

ITERATION NO. 6

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